Gram-schmidt algorithm from wednesday...

$\|t_1\| = \sqrt{t_1 \cdot t_1}$

$t_1 = a_1 \qquad q_1 = \dfrac{t_1}{\boxed{\|t_1\|}}$ of t_1

$t_2 = a_2 - \boxed{(a_2 \cdot q_1)}\, q_1 \simeq a_2 - \dfrac{1}{\|t_1\|}\dfrac{1}{\|t_1\|}(a_2 \cdot t_1)\, t_1 = a_2 - \dfrac{a_2 \cdot t_1}{t_1 \cdot t_1}\, t_1$

$\underbrace{\|t_1\|\,\|t_1\|}_{t_1 \cdot t_1}$

some books write like this ... (maybe ours)

claim t_2 is perpendicular to q_1

$t_2 \cdot q_1 = \left(a_2 - (a_2 \cdot q_1)\, q_1\right) \cdot q_1$

Then $q_1 \cdot q_1 = 1$

$= a_2 \cdot q_1 - (a_2 \cdot q_1)(q_1 \cdot q_1)$

$= a_2 \cdot q_1 - (a_2 \cdot q_1) = 0 \qquad q_2 = \dfrac{t_2}{\|t_2\|}$

$t_3 = a_3 - (a_3 \cdot q_1)\, q_1 - (a_3 \cdot q_2)\, q_2 \qquad q_3 = \dfrac{t_3}{\|t_3\|}$

$\vdots$

$t_n = a_n - (a_n \cdot q_1)\, q_1 - \cdots - (a_n \cdot q_{n-1})\, q_{n-1} \qquad q_n = \dfrac{t_n}{\|t_n\|}$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \qquad R = \begin{bmatrix} \|t_1\| & q_1 \cdot a_2 & q_1 \cdot a_3 & \cdots & q_1 \cdot a_n \\ & \|t_2\| & q_2 \cdot a_3 & \cdots & q_2 \cdot a_n \\ & & \ddots & & \vdots \\ & & & \ddots & q_{n-1} \cdot a_n \\ & & & & \|t_n\| \end{bmatrix}$$

$$QR = \begin{bmatrix} \vert & \vert & \cdots & ? \end{bmatrix} \qquad Q \begin{bmatrix} q_1 \cdot a_n \\ q_2 \cdot a_n \\ \vdots \\ q_{n-1} \cdot a_n \\ \|t_n\| \end{bmatrix} = (q_1 \cdot a_n)\, q_1 + (q_2 \cdot a_n)\, q_2 + \cdots$$

$$\cdots + (q_{n-1} \cdot a_n)\, q_n + \|t_n\|\, q_n$$

Example:

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

orthonormal columns

upper triangular

Make $A = QR$

$$t_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \qquad \|t_1\| = \sqrt{3^2+1^2+1^2+3^2} = \sqrt{20} = 2\sqrt{5} \qquad q_1 = \frac{1}{2\sqrt{5}}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}$$

$$t_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \cdot \frac{1}{\sqrt{20}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{20}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{1}{\sqrt{20}}\frac{1}{\sqrt{20}} \left(\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

dot product

-15

1

-5

-21

-40

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{1}{20} \left(\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{t_2}{\|t_2\|} = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$t_3 = \begin{bmatrix}1\\1\\-2\\8\end{bmatrix} - \left(\begin{bmatrix}1\\1\\-2\\8\end{bmatrix} \cdot \frac{1}{2\sqrt{5}}\begin{bmatrix}3\\1\\-1\\3\end{bmatrix}\right) \frac{1}{2\sqrt{5}}\begin{bmatrix}3\\1\\-1\\3\end{bmatrix}$$

$$- \left(\begin{bmatrix}1\\1\\-2\\8\end{bmatrix} \cdot \frac{1}{\sqrt{20}}\begin{bmatrix}1\\3\\3\\-1\end{bmatrix}\right) \frac{1}{\sqrt{20}}\begin{bmatrix}1\\3\\3\\-1\end{bmatrix}$$

dot products

$$\begin{array}{r}3\\1\\2\\24\\\hline 30\end{array} \qquad \begin{array}{r}1\\3\\-6\\-8\\\hline -10\end{array}$$

$$= \begin{bmatrix}1\\1\\-2\\8\end{bmatrix} - \frac{30}{20}\begin{bmatrix}3\\1\\-1\\3\end{bmatrix} - \frac{-10}{20}\begin{bmatrix}1\\3\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\-2\\8\end{bmatrix} - \frac{1}{2}\begin{bmatrix}9-1\\3-3\\-3-3\\9+1\end{bmatrix} = \begin{bmatrix}1\\1\\-2\\8\end{bmatrix} - \frac{1}{2}\begin{bmatrix}8\\0\\-6\\10\end{bmatrix}$$

$$= \begin{bmatrix}-3\\1\\1\\3\end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{20}}\begin{bmatrix}-3\\1\\1\\3\end{bmatrix}$$

Therefore

$$Q = \left[\, q_1 \,\middle|\, q_2 \,\middle|\, q_3 \,\right] \qquad R = \begin{bmatrix} ? \end{bmatrix}.$$