We have jumped ahead all the way to chapter 6.4 and will definitely return to chapter 5 later. There are two important algorithms in this class

- Gaussian elimination (row operations)
- Gram-Schmidt orthogonalization (column operations)

and I wanted to cover the second algorithm sooner than later, so there was more time for it to sink in.

an alternative way to write the Gram-Schmidt algorithm from class

$$\|t_1\| = \sqrt{t_1 \cdot t_1}$$

$$\|t_1\|^2 = t_1 \cdot t_1$$

substitute q_1 into the definition of t_2 and then perform all the normalization steps at the end.

$$t_1 = a_1 \qquad q_1 = \frac{t_1}{\|t_1\|}$$

$$t_2 = a_2 - \boxed{(q_1 \cdot a_2)}\, q_1 = a_2 - \left(\frac{t_1}{\|t_1\|} \cdot a_2\right) \frac{t_1}{\|t_1\|}$$

claim t_2 is perpendicular to q_1

so $q_1 \cdot q_1 = 1$

$$t_2 \cdot q_1 = \left(a_2 - (q_1 \cdot a_2)\, q_1\right) \cdot q_1$$

$$= a_2 \cdot q_1 - (q_1 \cdot a_2)\,\underbrace{(q_1 \cdot q_1)}_{\text{unit vector}}$$

$$\boxed{t_2 = a_2 - \frac{t_1 \cdot a_2}{t_1 \cdot t_1}\, t_1}$$

another way to write the same algorithm. Then create the unit vectors at the end...

Example of Gram-Schmidt

matrix with orthonormal columns

upper triangular matrix.

Goal is to factor $\underset{4\times 3}{A} = \underset{4\times 3}{Q}\,\underset{3\times 3}{R}$

factor this matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

In general $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ since the columns are independent

$$\underset{m\times n}{A} = \underset{m\times n}{Q}\;\underset{n\times n}{R}$$

Example ...

$$t_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad \|t_1\| = \sqrt{1+9+1+1} = \sqrt{12} \qquad q_1 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

don't do this

dot prod goes in R

$$t_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \boxed{\left(\frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}\right)} \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{1}{12}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}\right)\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

dot product

$$\begin{array}{r} -6 \\ -24 \\ -2 \\ -4 \\ \hline -36 \end{array}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$t_3 = \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix}\cdot\begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$\frac{30}{12}\cdot\frac{3\cdot5\cdot2}{3\cdot2\cdot2}$$

dot products

$$\begin{array}{r}-6\\9\\6\\-3\\\hline 6\end{array}$$

dot prod

$$\begin{array}{r}18\\3\\6\\3\\\hline 30\end{array}$$

$$= \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \underbrace{\frac{1}{12}6}_{\frac{1}{2}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \underbrace{\frac{1}{12}30}_{5/2}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \frac{1}{2}\left(\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} + \begin{bmatrix}15\\5\\5\\-5\end{bmatrix}\right)$$

$$= \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \begin{bmatrix}7\\4\\3\\-2\end{bmatrix} = \begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \qquad R = \begin{bmatrix} & ? & \end{bmatrix}.$$