Continue the example from last time $\underset{4\times 3}{A} = \underset{4\times 3}{Q}\,\underset{3\times 3}{R}$

$$q_1 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$Q = \left[\, q_1 \;\middle|\; q_2 \;\middle|\; q_3 \,\right] = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$$R = \begin{bmatrix} \|t_1\| & q_1\cdot a_2 & q_1\cdot a_3 & \cdots & q_1\cdot a_n \\ & \|t_2\| & \boxed{q_2\cdot a_3} & & q_2\cdot a_n \\ & 0 & \ddots & & \vdots \\ & & & \ddots & q_{n-1}\cdot a_n \\ & & & & \|t_n\| \end{bmatrix} = \begin{bmatrix} \|t_1\| & q_1\cdot q_2 & q_1\cdot q_3 \\ 0 & \|t_2\| & q_2\cdot q_3 \\ 0 & 0 & \|t_3\| \end{bmatrix}$$

Note R is invertible because $\|t_i\|$'s are non-zero, which was because A had linearly ind. columns

$$\|t_1\| = \sqrt{1+9+1+1} = \sqrt{12}$$

$$\|t_2\| = \sqrt{12}$$

$$\|t_3\| = \sqrt{12}$$

$\leftarrow$ diagonal of R

$$q_1 \cdot a_2 = \left(\frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \right) = \frac{1}{\sqrt{12}} \cdot (-36)$$

dot product

$$\begin{array}{r} -6 \\ -24 \\ -2 \\ -4 \\ \hline -36 \end{array}$$

$$q_1 \cdot a_3 = \left(\frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right) = \frac{1}{\sqrt{12}} \cdot 6$$

$$\begin{array}{r} -6 \\ 9 \\ 6 \\ -3 \\ \hline 6 \end{array}$$

$$q_2 \cdot a_3 = \left(\frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right) = \frac{1}{\sqrt{12}} \cdot 30$$

dot prod

$$\begin{array}{r} 18 \\ 3 \\ 6 \\ 3 \\ \hline 30 \end{array}$$

$$R = \begin{bmatrix} \sqrt{12} & -36/\sqrt{12} & 6/\sqrt{12} \\ 0 & \sqrt{12} & 30/\sqrt{12} \\ 0 & 0 & \sqrt{12} \end{bmatrix} = \frac{6}{\sqrt{12}} \begin{bmatrix} 2 & -6 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$R = \frac{6}{\sqrt{12}} \begin{bmatrix} 2 & -6 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix} \quad \frac{1}{2} \begin{bmatrix} -2 & 12 & 12 \\ 6 & -16 & 6 \\ 2 & -4 & 12 \\ 2 & -8 & -6 \end{bmatrix}$$

$$QR = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$ ☑ okay...

What to do with the $A = QR$ factorization?

Recall Q is a matrix with orthonormal columns

$$Q = \left[q_1 \middle| q_2 \middle| q_3 \right]$$ Thus $$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ \hline q_2^T \\ \hline q_3^T \end{bmatrix} \left[q_1 \middle| q_2 \middle| q_3 \right] = \begin{bmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & q_1 \cdot q_3 \\ q_2 \cdot q_1 & q_2 \cdot q_2 & q_2 \cdot q_3 \\ q_3 \cdot q_1 & q_3 \cdot q_2 & q_3 \cdot q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Therefore $\boxed{\underset{3\times 4}{Q^T}\underset{4\times 3}{Q} = I} \in \mathbb{R}^{3\times 3}$

$A = QR$

Idea... we usually try to solve $Ax = b$

$\boxed{Ax = b}$

$QRx = b$ mult both sides by Q^T

$Q^TQRx = Q^Tb$

$IRx = Q^Tb$

$\boxed{Rx = Q^Tb}$

Consequence of the QR factorization...

Since R is upper triangular don't need to do any elimination steps to solve this equation...

Thus:

If $Ax = b$ then $\underbrace{Rx = Q^Tb}$

This equation is easy to solve for x...

Therefore, if $Ax = b$ had a solution you could find it by solving $Rx = Q^Tb$

Since R is invertible, $Rx = Q^Tb$ always has a solution no matter what b is...

On the other hand... since $A \in \mathbb{R}^{4\times 3}$

$Ax = b$ there is not a pivot in each row of the row echelon form of A so there are many values of b for which this equation is inconsistent..

When $Ax=b$ doesn't have a solution

Then $Rx = Q^T b$ still has a solution
and that soln. is the least-square minimizer
for $Ax \approx b$...