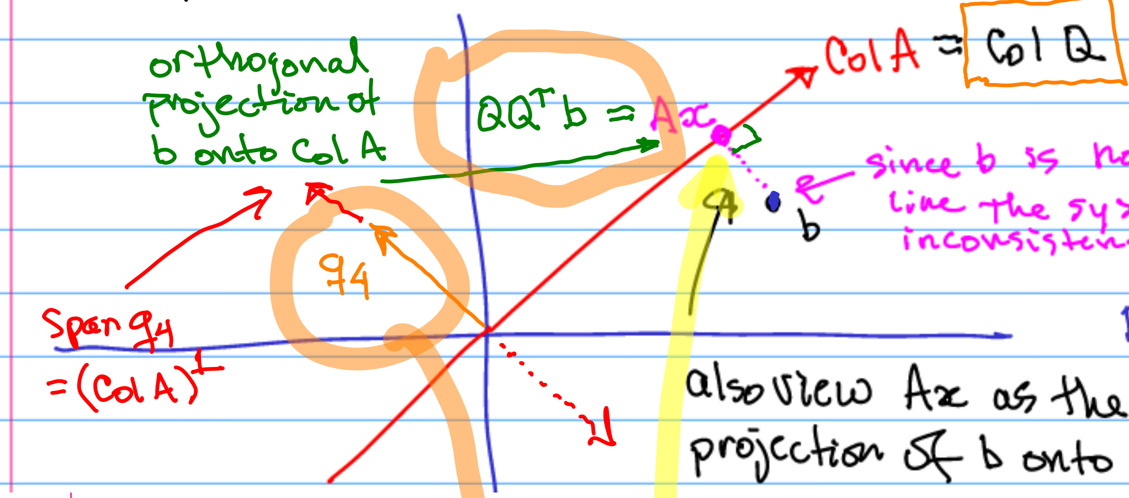
$A \in \mathbb{R}^{4\times 3}$, $A = QR$

From last time



also view Ax as the perpendicular projection of b onto Col A.

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

The q_i's are orthonormal

$$q_i \cdot q_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

This defines the closest point Ax in Col A to b. $\rightarrow$

The green square means that the dot product of $Ax - b$ (That's the vector corresponding to the pink dotted line) is zero when dotted into any vector in Col A.

Thus

$$(Ax - b) \cdot z = 0$$

for every $z \in \mathrm{Col}\,A$.

What's QQ^Tb ?

$Q \in \mathbb{R}^{4\times 3}$

$Q^TQ = I$ but QQ^T is not the identity unless Q is square...

$$\underbrace{QQ^Tb}_{\text{orthogonal projection onto Col}A} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \end{bmatrix} b = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1 \cdot b \\ q_2 \cdot b \\ q_3 \cdot b \end{bmatrix}$$

add another vector q_4 to make Q square...

$$= (q_1 \cdot b)q_1 + (q_2 \cdot b)q_2 + (q_3 \cdot b)q_3 + (q_4 \cdot b)q_4 = b$$

may not be b since Q was not square...

adding q_4 makes Q square so then $QQ^Tb = b$

leaving out the q_4 results in a point in Col A that equal to

$b - (q_4 \cdot b)q_4$

so it satisfies the orthogonality needed to be the closest point in Ax to b.

EM 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$\boxed{(\text{Row } A)^\perp = \text{Nul } A} \quad \text{and} \quad \boxed{(\text{Col } A)^\perp = \text{Nul } A^T}$$

What do all these terms mean

are the same once one knows Col A^T = Row A

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$$\text{Nul } A = \{ x : Ax = 0 \} \subseteq \mathbb{R}^n$$

$$A = \begin{bmatrix} r_1^T \\ \hline r_2^T \\ \hline \vdots \\ \hline r_m^T \end{bmatrix} \in \mathbb{R}^{m \times n} \qquad \text{Row } A = \text{Span}\{r_1, r_2, \dots, r_m\}$$

$$= \{c_1 r_1 + c_2 r_2 + \dots + c_m r_m : c_i \in \mathbb{R}\}$$

$$A^T = \left[\, r_1 \mid r_2 \mid \dots \mid r_m \,\right] \in \mathbb{R}^{n \times m} \qquad \text{Col } A^T = \{ A^T c : c \in \mathbb{R}^m \}$$

$$= \{c_1 r_1 + c_2 r_2 + \dots + c_m r_m : c_i \in \mathbb{R}\}$$

Therefore $\text{Col } A^T = \text{Row } A$

$$(\text{Row } A)^\perp = \text{Nul } A$$

$$(\text{Col } A^T)^\perp = \text{Nul } A$$

Substitute $B = A^T$, then $B^T = A$

$$\boxed{(\text{Col } B)^\perp = \text{Nul } B^T}$$

$$(\text{Col } A)^\perp = \text{Nul } A^T$$

Suppose H is a subspace (either Col A or something)

Then

$$H^{\perp} = \{ x : x \cdot y = 0 \text{ for all } y \in H \}.$$