$A \in \mathbb{R}^{m\times n}$ $\quad A^T \in \mathbb{R}^{n \times m}$

EM 3 Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad \boxed{(\text{Col } A)^{\perp} = \text{Nul } A^T}$$

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$$\text{Nul } A^T = \{ y : A^T y = 0 \} \subseteq \mathbb{R}^m$$

$$H^{\perp} = \{ z : z \cdot w = 0 \text{ for all } w \in H \}$$

$w \in \text{Col } A$ means $w = Ax$ for some $x \in \mathbb{R}^n$

$$
\begin{aligned}
(\text{Col } A)^{\perp} &= \{ z : z \cdot w = 0 \text{ for all } w \in \text{Col } A \} \\
&= \{ z : z \cdot Ax = 0 \text{ for all } x \in \mathbb{R}^n \} \\
&= \{ z : Ax \cdot z = 0 \text{ for all } x \in \mathbb{R}^n \} \\
&= \{ z : (Ax)^T z = 0 \text{ for all } x \in \mathbb{R}^n \} \\
&= \{ z : x^T A^T z = 0 \text{ for all } x \in \mathbb{R}^n \} \\
&= \{ z : x \cdot A^T z = 0 \text{ for all } x \in \mathbb{R}^n \}
\end{aligned}
$$

Claim $A^T z = 0$ ~~if and only if~~ is equivalent to $x \cdot A^T z = 0$ for all $x \in \mathbb{R}^n$

If $x \cdot A^T z = 0$ for all $x \in \mathbb{R}^n$ then it must be zero for a particular $x = A^T z$. Then

Substitute $A^T z \cdot A^T z = 0$ which is the same as $\| A^T z \|^2 = 0$ which means $A^T z = 0$.

$(\mathrm{Col}\,A)^{\perp} = \{z : x \cdot A^T z = 0 \text{ for all } x \in \mathbb{R}^n\}$

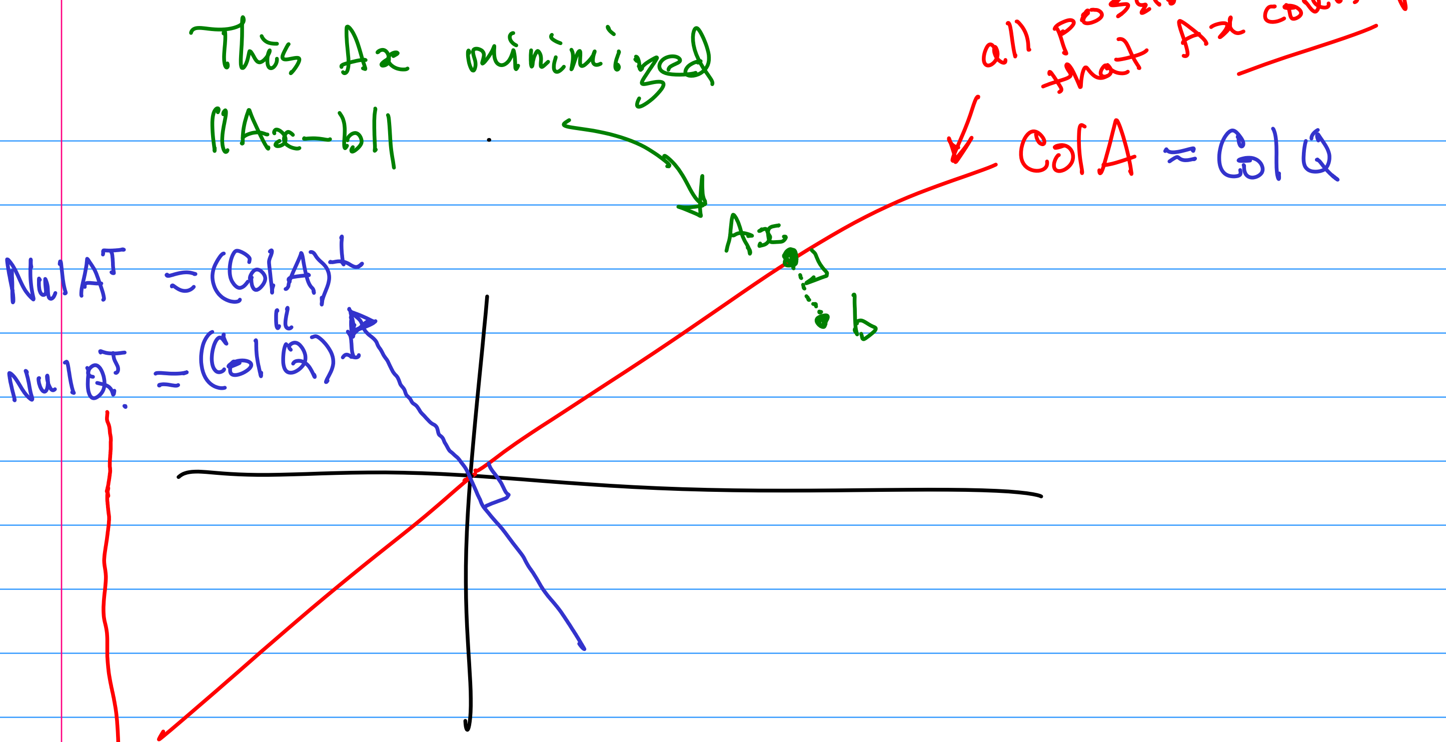
$\approx \{z : A^T z = 0\} = \mathrm{Nul}\,A^T$

related to the normal equations for solving $Ax=b$ using least squares minimization...

Recall If $A \in \mathbb{R}^{m \times n}$ and $m > n$ then we have more equations in $Ax=b$ than there are variables. This means that $Ax=b$ is inconsistent most of the time. Even if $Ax=b$ were consistent trying to solve it on a computer would introduce just a little rounding error that again makes it inconsistent...

Idea is to minimize $\|Ax-b\|$

We did this geometrically



How to find x?

$$A = QR$$

$$Ax = b$$

$$QRx = b$$

$$Q^TQRx = Q^Tb$$

Solve this for x → $\boxed{Rx = Q^Tb}$ ← do it with numbers...

Normal equations:

$$Ax = b$$

solve this for x → $\boxed{A^TAx = A^Tb}$ ← need to understand this more...

least-squares solution of $A\mathbf{x} = \mathbf{b}$

15. $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}}_{Q} \overbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}^{R}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

Solve $Ax = b$ in the least squares sense

$Rx = Q^T b$

$$Q^T b = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 21 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\begin{array}{r} 14 \\ +6 \\ 1 \\ \hline \end{array} \qquad \begin{array}{r} -7 \\ 6 \\ -2 \\ \hline \end{array}$$

$$\underbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}_{R}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 7 \\ -1 \end{bmatrix}}_{Q^T b}$$

Thus

$x_2 = -1$

$x_1 = \frac{1}{3}(7 - 5x_2) = \frac{1}{3}(7+5) = \frac{12}{3} = 4$

Answer: $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ This x minimizes $\|Ax - b\|$.