output $\quad$ input $\quad$ input

$A \in \mathbb{R}^{m \times n}$ $\qquad\qquad$ $A^T \in \mathbb{R}^{n \times m}$

M 3 Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

equivalent

$$\boxed{(\text{Row } A)^{\perp} = \text{Nul } A} \quad \text{and} \quad \boxed{(\text{Col } A)^{\perp} = \text{Nul } A^T}$$

$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

If $w \in \text{Col } A$ that means $w = Ax$ for some $x \in \mathbb{R}^n$

$\text{Nul } A^T = \{ y : A^T y = 0 \} \subseteq \mathbb{R}^m$

$H^{\perp} = \{ z : z \cdot w = 0 \text{ for all } w \in H \}$

$(\text{Col } A)^{\perp} = \{ z : z \cdot w = 0 \text{ for all } w \in \text{Col } A \}$

$\{ z : z \cdot Ax = 0 \text{ for all } x \in \mathbb{R}^n \}$

$\{ z : x \cdot A^T z = 0 \text{ for all } x \in \mathbb{R}^n \}$

Note: all mean the same thing.
$$\left\{ \begin{aligned} & z \cdot Ax = 0 \\ & Ax \cdot z = 0 \\ & (Ax)^T z = 0 \\ & x^T A^T z = 0 \\ & x \cdot A^T z = 0 \end{aligned} \right.$$

Claim: $\boxed{A^T z = 0}$ if and only if $\boxed{x \cdot A^T z = 0 \text{ for all } x \in \mathbb{R}^n}$

is equivalent to

means the same thing as

"$\Rightarrow$" $A^T z = 0$ implies $x \cdot A^T z = 0$ for all $x \in \mathbb{R}^n$ is easy...

"$\Leftarrow$" Suppose $x \cdot A^T z = 0$ for all $x \in \mathbb{R}^n$. Why is $A^T z = 0$?

what vector $A^T z$ could be perpendicular to all other vectors?

what vector is perpendicular to itself? $\quad x = A^T z$ and plug it in..

$A^T z \cdot A^T z = 0$ which means $\|A^T z\|^2 = 0$ so $A^T z = 0$.

Therefore

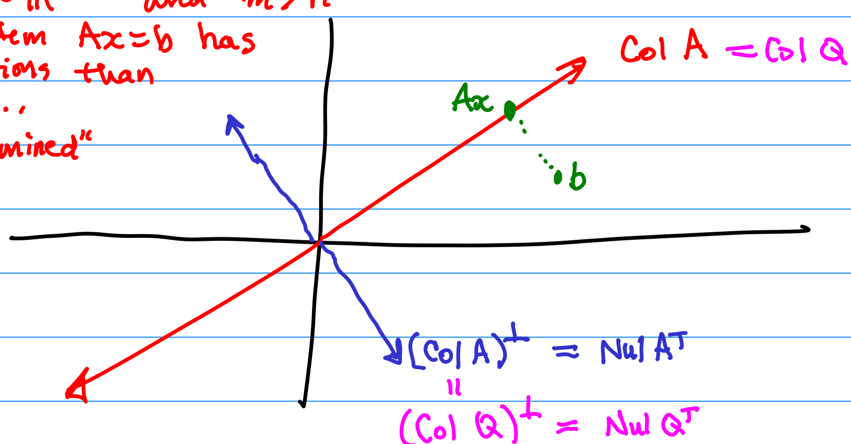
$(\text{Col } A)^{\perp} = \{ z : x \cdot A^T z = 0 \text{ for all } x \in \mathbb{R}^n \}$

$= \{ z : A^T z = 0 \} = \text{Nul } A^T$.

Let $A = QR$

Consider $A \in \mathbb{R}^{m \times n}$ and $m > n$
so the system $Ax = b$ has more equations than unknowns..
"overdetermined"

Idea: minimize $\|Ax - b\|$



What's left is to solve some least squares problems:

$Ax=b$ overdetermined, factor $A=QR$

$QRx=b$

$Q^TQRx=Q^Tb$

$IRx=Q^Tb$

$\boxed{Rx=Q^Tb}$ ← solve this for x and that x turns out to minimize $\|Ax-b\|$.

EXAMPLE

$\in \mathbb{R}^{4\times 2}$ so $Ax=b$ is overdetermined

16. $A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \overbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}}^{R}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$

Solve $Rx=Q^Tb$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 17 \\ 9 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

$$\begin{array}{r} {}^{3}17 \\ 5 \\ \hline 45 \\ -27 \\ \hline 58 \end{array}$$

$$\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

$2x_1+3x_2 = 17/2$

$5x_2 = 9/2$

$x_2 = \frac{9}{10}$ $\quad x_1 = \frac{1}{2}\left(\frac{17}{2} - 3x_2\right) = \frac{1}{2}\left(\frac{17}{2} - \frac{27}{10}\right) = \frac{1}{2}\,\frac{58}{10} = \frac{29}{10}$

Thus $x = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$ minimizes $\|Ax-b\|$.