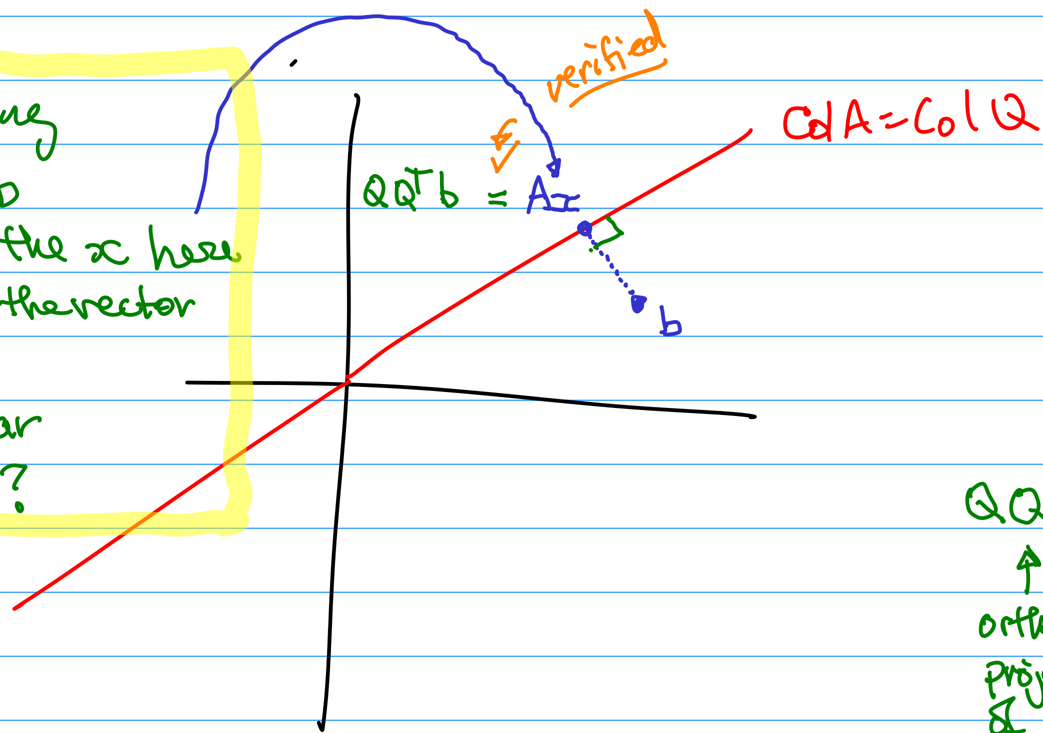
least-squares solution of $Ax = b$.

$$\textbf{15.}\quad A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}}_{Q} \overbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}^{R},\ \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Solve $Ax = b$ in the least squares sense

$Rx = Q^T b$

Does solving $Rx = Q^T b$ really give the x here that makes the vector $Ax - b$ perpendicular to Col A ?



$QQ^T b$ ← orthogonal projection of b

$A \in \mathbb{R}^{m \times n}$ where $m \geq n$ and columns of A are linearly independent.

Factor A using Gram-Schmidt algorithm

$$\underset{m\times n}{A} = \underset{m\times n}{Q}\,\underset{n\times n}{R}$$

Q ← matrix with orthonormal columns

R ← upper triangular invertible matrix

Then we solved $Rx = Q^Tb$ and claimed this minimized $\|Ax-b\|$.

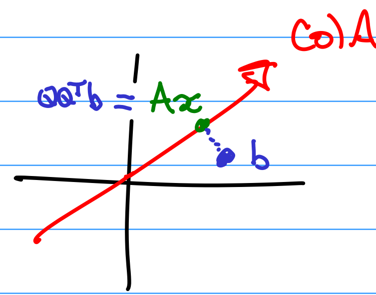
Check that the geometry in the picture is really what the algebra accomplished...

Since $Rx = Q^Tb$

Solve for x. Since R is invertible use R^{-1} theoretically...

$$x = R^{-1}Q^Tb$$

Plug it it $\quad Ax = AR^{-1}Q^Tb = QQ^Tb$



☑ Check the perpendicularity condition:

$$(Ax-b)\cdot y = 0 \quad \text{for every } y \in \text{Col } A$$

Check

$$(AR^{-1}Q^Tb - b)\cdot y = 0 \quad \text{for every } y \in \text{Col } A$$

Remember $A = QR$. Need to show

$$\boxed{(Q\cancel{R}\cancel{R^{-1}}Q^Tb - b)}\cdot y = 0 \quad \text{for every } y \in \underset{\text{Col } Q}{\underset{\shortparallel}{\text{Col } A}}$$

Need to show

$$(QQ^T b - b)\cdot y = 0 \quad \text{for every } y \in \operatorname{Col} Q$$

since $\operatorname{Col} Q = \operatorname{Span}\{q_1, q_2, \dots, q_n\}$

Need to show

$$\boxed{(QQ^T b - b)\cdot q_k = 0} \quad \text{for every } q_k$$

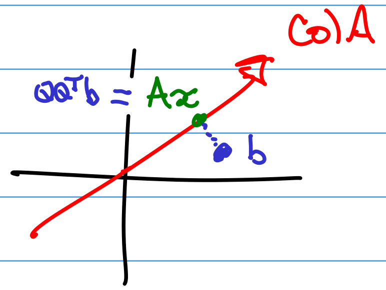
$$QQ^T b = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}\begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} b = q_1(q_1\cdot b) + q_2(q_2\cdot b) + \dots + q_n(q_n\cdot b)$$

$$QQ^T b\cdot q_k = \big(q_1(q_1\cdot b) + q_2(q_2\cdot b) + \dots + q_n(q_n\cdot b)\big)\cdot q_k = (q_k\cdot q_k)(q_k\cdot b)$$
$$= q_k\cdot b$$

On the other hand

$$b\cdot q_k = q_k\cdot b$$

Thus $(QQ^T b - b)\cdot q_k = (QQ^T b\cdot q_k - b\cdot q_k) = q_k\cdot b - q_k\cdot b = 0$



this verifies the perdicularity shown in the graph and so

$\|Ax - b\|$ is minimized by solving

$$\boxed{Rx = Q^T b.}$$

Connect this with the normal equations:

Claim: One can also find the minimizing x by solving $\boxed{A^T A x = A^T b}$.

Recall $A = QR$ plug it in ...

$$(QR)^T QR x = (QR)^T b$$

$$R^T Q^T Q R x = R^T Q^T b$$

$$\cancel{(R^T)^{-1}} \cancel{R^T} Q^T Q R x = \cancel{(R^T)^{-1}} \cancel{R^T} Q^T b$$

$$\cancel{Q^T Q}^{=I} R x = Q^T b$$

$$\boxed{Rx = Q^T b}$$

Since R is invertible claim R^T is also invertible...

$Q^T Q = I$ since the columns of Q are orthonormal

Claim R^T is invertible, that is need to show that $(R^T)^{-1}$ exists...

Let $\boxed{B = (R^{-1})^T}$. Then it's enough to show that $R^T B = I$

because that would mean $B = (R^T)^{-1}$.

since B is square and R^T is square...

Thus

$$R^T B = R^T (R^{-1})^T = (R^{-1} R)^T = I^T = I$$

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

blue indicates fixes for arithmetic errors from in class

Solve this

$$\begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix} x = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

finally

The answer from last time was

$$x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Same indicates that both $A^T A x = A^T b$ and $Rx = Q^T b$ have the same solutions...

Check:

$$\begin{bmatrix} 9 & 15 \\ 15 & 26 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 36 - 15 \\ 60 - 26 \end{bmatrix} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$