Sept 17, 2001

from last time...

$$U = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix} = LU.$$

Want to solve $Ax = b$ where

$$A = \begin{bmatrix} 2 & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

How? Using $A = LU$ factorizations...

$Ax = b$

$LUx = b$

$y = Ux$

Thus

$$\begin{cases} Ly = b \\ Ux = y \end{cases}$$

Note this is just usual algebra stuff (i.e. treating letters as numbers) except one has to remember that LU is composition of function so the order matters! Don't write it as UL by accident...

pivots are all 1's for L...

$$\begin{bmatrix} 1 & 0 & 0 \\ y_2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

$y_1 = 2$

$y_2 = -1 - \frac{1}{2} y_1 = -1 - 1 = -2$

$y_3 = 7$

$-3y_1 - 2y_2 + y_3 = 5$

$y_3 = 5 + 3y_1 + 2y_2 = 5 + 3(2) + 2(-2) = \square$

Now solve $Ux = y$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 7 \end{bmatrix}$$

missing pivot → can't solve for x_3 ⇒ no solution...

Interesting example --

How about $Ax = b$ where $b = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ y_2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$y_1 = 2$

$y_2 = -1 - \frac{1}{2} y_1 = -1 - 1 = -2$

$y_3 = 0$

$-3y_1 - 2y_2 + y_3 = 5$

$y_3 = -2 + 3y_1 + 2y_2 = -2 + 3(2) + 2(-2) = \boxed{0}$

Now continue...

P P F

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & \boxed{7} & -5 \\ 0 & 0 & \boxed{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

missing pivot

$x_1 = \frac{1}{2}(2 + 4x_2 - 2x_3) = \frac{3}{7} - \frac{3}{7}x_3$

$x_2 = \frac{1}{7}(-2 + 5x_3) = \frac{-2}{7} + \frac{5}{7}x_3$

$x_3 = x_3$

$x_1 = \frac{1}{2}(2 + 4x_2 - 2x_3) = \frac{1}{2}\left(2 + 4\left(-\frac{2}{7} + \frac{5}{7}x_3\right) - 2x_3\right)$

$= 1 - \frac{4}{7} + \left(\frac{10}{7} - 1\right)x_3 = \frac{3}{7} + \frac{3}{7}x_3$

Answer:

$$x = \begin{bmatrix} \frac{3}{7} - \frac{3}{7}x_3 \\ -\frac{2}{7} + \frac{5}{7}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -2/7 \\ 0 \end{bmatrix} + \begin{bmatrix} +3/7 \\ 5/7 \\ 1 \end{bmatrix} x_3$$

P + green "plane" actually a line...