Row operations: Building blocks of Gaussian Elimination

1. Elimination steps $r_i \leftarrow r_i - \alpha r_j \quad (i \neq j)$
2. Row swaps $r_i \leftrightarrow r_j$
3. Scaling operation $r_i \leftarrow \alpha r_i$

Claim each of these operations are linear...

What does it mean for a function f to be linear?

1. $f(x+y) = f(x) + f(y)$
2. $f(\alpha x) = \alpha f(x)$.

Connection between Matrix-Matrix multiplication and Matrix-vector mult.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & -2 \end{bmatrix} = \left[\, b_1 \mid b_2 \mid b_3 \,\right]$$

divide B into columns

vectors

$$AB = A\left[\, b_1 \mid b_2 \mid b_3 \,\right] = \left[\, Ab_1 \mid Ab_2 \mid Ab_3 \,\right]$$

$f(x) = Ax \quad f(b_1) = Ab_1, \ldots, f(b_3) = Ab_3$

$$AB = \left[f(b_1) \mid f(b_2) \mid f(b_3) \right] = f(B)$$

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① Elimination steps $\quad r_i \leftarrow r_i - \alpha r_j \quad (i \neq j)$

Suppose $f(A)$ represents the process of performing the row operation

$r_2 \leftarrow r_2 + 3r_1$

Claim this is linear:

$$f(A) = f\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

$$f(B) = f\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -1 & 6 \\ 4 & 1 & -2 \end{bmatrix}$$

$$f(A) + f(B) = \boxed{\begin{bmatrix} 2 & 2 & 4 \\ 12 & 10 & 21 \\ 11 & 9 & 7 \end{bmatrix}}$$

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 6 & 4 & 9 \\ 11 & 9 & 7 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 3r_1$

$$f(A+B) = f\left(\begin{bmatrix} 2 & 2 & 4 \\ 6 & 4 & 9 \\ 11 & 9 & 7 \end{bmatrix}\right) = \boxed{\begin{bmatrix} 2 & 2 & 4 \\ 12 & 10 & 21 \\ 11 & 9 & 7 \end{bmatrix}}$$

Therefore $f(A) + f(B) = f(A+B)$ ...

Similarly $f(\alpha A) = \alpha f(A)$ and so f is a linear function...

In fact any row operation, e.g. swapping rows or rescaling them works the same way...

Consequence:

Since every row operation corresponds to a linear function

& Every linear function corresponds to matrix multiplication...

Then

every row operation corresponds to matrix multiplication...

Question what are those matrices?

Example f is the elimination step $r_2 \leftarrow r_2 + 3r_1$

Let M be the matrix corresponding to f:

Thus, $f(x) = Mx$

$$M = \left[\begin{array}{c|c|c|c} f(e_1) & f(e_2) & \cdots & f(e_n) \end{array}\right]$$

$$= f\left(\left[\begin{array}{c|c|c|c} e_1 & e_2 & \cdots & e_n \end{array}\right]\right)$$

$$= f\left(\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 0 \\ & 0 & & 1 \end{bmatrix}\right) = f(I)$$

Therefore the matrix corresponding to $r_2 \leftarrow r_2 + 3r_1$ is

$$M = f(I) = f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notation

$$[r_2 \leftarrow r_2 + 3r_1] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{21}(3)$

Check...

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 3r_1$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 3r_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

② Try $r_2 \leftrightarrow r_3$ (swap rows)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_2 \leftrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = M$$

③ Rescaling

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_2 \leftarrow \tfrac{1}{2} r_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5/2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Notation

$$\left[r_2 \leftarrow \tfrac{1}{2} r_2 \right] \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$