This set, called the **zero subspace**, also satisfies the conditions for a subspace.

Column Space and Null Space of a Matrix

Talking:

Subspaces of $\mathbb{R}^n$ usually occur in applications and theory in one of two ways. In both cases, the subspace can be related to a matrix.

$$A \in \mathbb{R}^{\overset{\text{range}}{m} \times \overset{\text{domain...}}{n}}$$

$$\text{range } f = \{ f(x) : x \in \mathbb{R}^n \}$$

DEFINITION

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

Same as the range of f where $f(x) = Ax$.

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

are in the domain

$Ax \in \mathbb{R}^m$

If $A = [\mathbf{a}_1 \;\; \cdots \;\; \mathbf{a}_n]$, with the columns in $\mathbb{R}^m$, then Col A is the same as Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Example 4 shows that the **column space of an** $\boldsymbol{m \times n}$ **matrix is a subspace of** $\mathbb{R}^m$. Note that Col A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, Col A is only part of $\mathbb{R}^m$.

DEFINITION

The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

" such that "

$$\text{Nul } A = \{ x \in \mathbb{R}^n : Ax = 0 \}.$$

elements in the set

condition those elements have to satisfy.

THEOREM 13

The pivot columns of a matrix A form a basis for the column space of A.

algorithm for finding a basis...

$$A = \left[v_1 \,\middle|\, v_2 \,\middle|\, v_3 \,\middle|\, v_4 \,\middle|\, v_5 \,\middle|\, v_6 \,\middle|\, v_7 \,\middle|\, v_8 \,\middle|\, v_9 \right]$$

Gaussian Elimination to obtain row echelon form

$$\begin{array}{ccccccccc} F & F & P & F & P & P & F & F & F \end{array}$$
$$\begin{bmatrix} 0 & 0 & * & \cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & * & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

9 columns...

$*$ mean non-zero entry...

The basis is $\{v_3, v_5, v_6\}$ the columns in the original matrix the correspond to the pivots...

EXAMPLE Find a basis of the column space:

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

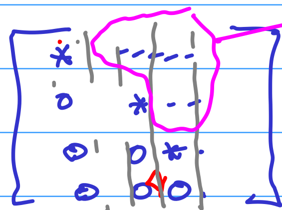
Since col A is already describe as the span of the "columns of A" the all that's needed is to identify which of those columns correspond to pivots

Suppose the 3^{rd}, 5^{th} and 6^{th} columns were pivots...

$$\text{Basis} = \{v_3, v_5, v_6\}$$

$$\text{Col } A = \{Bx : x \in \mathbb{R}^3\} \quad \text{where } B = \begin{bmatrix} v_3 & v_5 & v_6 \end{bmatrix}$$

If I perform Gauss elimin on B to find the row eschelon form



numbers are the same as in

exactly the same columns as were in the pivot columns for the row eschelon form of A.

What about a basis for

$$\text{Nul } A = \{x \in \mathbb{R}^n : Ax = 0\}$$

Example:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}$$ find a basis of Nul A.

Elimination steps.. $r_2 \leftarrow r_2 - 2r_1$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

rescale the rows $\quad r_1 \leftarrow \frac{1}{2} r_1$

lucky, usually would need another elimination step... $\quad r_2 \leftarrow \frac{1}{-3} r_2$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -4/3 \end{bmatrix}$$ ← Reduced row eschelon form...

Solve by substitution

$x_2 - \frac{4}{3}x_3 = 0, \quad x_2 = \frac{4}{3}x_3$

$x_1 - \frac{1}{2}x_3 = 0, \quad x_1 = \frac{1}{2}x_3$

Therefore

$$x = \begin{bmatrix} \frac{1}{2}x_3 \\ \frac{4}{3}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 4/3 \\ 1 \end{bmatrix} x_3$$

Basis right here...

$$\text{Nul } A = \{ Nx : x \in \mathbb{R}^1 \} \text{ where } N = \begin{bmatrix} 1/2 \\ 4/3 \\ 1 \end{bmatrix}.$$

We'll finish up chapter 2 next time and move on to chapter 3 and a discussion of determinants...