$2-\frac{2}{3}=\frac{6-2}{3}=\frac{4}{3}$

$9-\frac{5}{3}=\frac{27-5}{3}=\frac{22}{3}$

$6-\frac{2}{3}=\frac{18-2}{3}=\frac{16}{3}$

$-2+\frac{14}{3}=\frac{14-6}{3}=\frac{8}{3}$

$3+\frac{35}{3}=\frac{35+9}{3}=\frac{44}{3}$

$6+\frac{14}{3}=\frac{18+14}{3}=\frac{32}{3}$

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

Find Col A and Nul A

$r_2 \leftarrow r_2 + \frac{2}{3}r_1$

$r_3 \leftarrow r_3 + \frac{5}{3}r_1$

$r_4 \leftarrow r_4 + \frac{2}{3}r_1$

$$\begin{array}{r} 121 \\ -38 \\ \hline 83 \end{array}$$

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & 22/3 & 44/3 & 8 & 19 \\ 0 & 16/3 & 32/3 & 5 & 13 \end{bmatrix}$$

$r_3 \leftarrow r_3 - \frac{11}{2}r_2$

$r_4 \leftarrow r_4 - 4r_2$

$\frac{44}{3}-\frac{11}{2}\frac{8}{3}=\frac{88-88}{6}=0$ lucky

$8-\frac{11}{2}9=\frac{16-99}{2}=-\frac{83}{2}$

$19-\frac{11}{2}11=\frac{38-121}{2}=-\frac{83}{2}$

$\frac{32}{3}-4\frac{8}{3}=0$ lucky.

$5-4\cdot 9=5-36=-31$

$13-4\cdot 11=13-44=-31$

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & 0 & 0 & -83/2 & -83/2 \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

$r_3 \leftarrow \frac{-2}{83}r_3$

$r_4 \leftarrow \frac{-1}{31}r_4$

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$r_4 \leftarrow r_4 - r_3$

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - 3r_3$

$r_2 \leftarrow r_2 - 9r_3$

$9-3=6$

$11-9=2$

$$\begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 4/3 & 8/3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_2 \leftarrow \frac{3}{2}r_2$

$$\begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow \frac{1}{3} r_1$

$r_2 \leftarrow \frac{1}{2} r_2$

$$\begin{bmatrix} 1 & -1/3 & 7/3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 + \frac{1}{3} r_2$

$\frac{7}{3} + \frac{1}{3} 2 = \frac{9}{3} = 3$

$2 + \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \frac{4+1}{2} = 5/2$

$$\begin{array}{ccccc} P & P & F & P & F \\ \boxed{1} & 0 & 3 & 0 & 5/2 \\ 0 & \boxed{1} & 2 & 0 & 3/2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

Col A has a basis $\underbrace{\left\{ \begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix} \right\}}_{\text{3 vectors in basis}}$

$$B = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 2 & 7 \\ -5 & 9 & 3 \\ -2 & 6 & 3 \end{bmatrix}$$

Col A = Col B

dim(Col A) = 3

Now the basis for the null space $Ax = 0$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ P & P & F & P & F \\ \boxed{1} & 0 & 3 & 0 & 5/2 \\ 0 & \boxed{1} & 2 & 0 & 3/2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$x_1 + 3x_3 + 5/2\, x_5 = 0$

$x_2 + 2x_3 + 3/2\, x_5 = 0$

$x_4 + x_5 = 0$

solve...

$x_4 = -x_5$

$x_2 = -2x_3 - 3/2\, x_5$

$x_1 = -3x_3 - 5/2\, x_5$

Solution to $Ax=0$

$$x = \begin{bmatrix} -3x_3 - 5/2\, x_5 \\ -2x_3 - 3/2\, x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Basis

The basis of Nul A is $\underbrace{\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}}_{\text{put the basis into a matrix...}}$

Nullspace matrix

$$N = \begin{bmatrix} -3 & -5/2 \\ -2 & -3/2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{5 \times 2}$$

Nul A = Span of basis

$= \{ Nx : x \in \mathbb{R}^2 \} = \text{Col}(N)$

Since 2 vectors in the basis for Nul(A)
then $\dim(\text{Nul}(A)) = 2$.

Section 2.9

Theorem: If $A \in \mathbb{R}^{\overset{\text{rows}}{m} \times \overset{\text{cols}}{n}}$ then

$$\underbrace{\dim(\text{Col } A)} + \dim(\text{Nul } A) = n$$ the number of columns in A.

Notation $\boxed{\text{rank } A} = \dim(\text{Col } A) = \#$ of pivots

definition of rank of A

Note also $\dim(\text{Nul } A) = \#$ of free vbls...

Definition of a basis of a space H

① Vectors are lin ind.

② Vectors span the space H.

Theorem if $\dim H = p$ then

any set of p ind. vectors $\Rightarrow$ they span H

any set of p vectors which span H $\Rightarrow$ they're independent