ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row r_i by the sum of itself and a multiple $-\alpha$ of another r_j row.[1] $i \neq j$
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

① $r_i \leftarrow r_i - \alpha r_j, \; i \neq j$ elimination step..
(subtracting one equation from another)

② $r_i \leftrightarrow r_j \quad i \neq j$ row swap

③ $r_i \leftarrow \alpha r_i \quad \alpha \neq 0$ scaling

Example

13.
$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

first step eliminate x_1 from the 2nd equation...

$\boxed{r_2 \leftarrow r_2 - 2r_1}$ ← elimination step...

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Augmented Matrix

right hand side

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

① $r_i \leftarrow r_i - \alpha r_j, \; i \neq j$
② $r_i \leftrightarrow r_j \quad i \neq j$
③ $r_i \leftarrow \alpha r_i \quad \alpha \neq 0$

- use row operations to create the echelon form of the matrix...

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \qquad \boxed{r_2 \leftarrow r_2 - 2r_1}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \qquad \boxed{r_3 \leftarrow r_3 - \tfrac{1}{2} r_2}$$

this 0 is unchanged

eliminate this x_2 in the third eq.

$$5 - \frac{15}{2} = \frac{10-15}{2} = -\frac{5}{2}$$

$$-2 + \frac{9}{2} = \frac{-4+9}{2} = \frac{5}{2}$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5/2 & 5/2 \end{array}\right] \end{array} \leftarrow \text{eshelon form...}$$

diagonal stripe for the eshelon...

Convert back to algebraic notation...

$$\begin{cases} x_1 \quad -3x_3 = 8 \\ 2x_2 + 15x_3 = -9 \\ -\tfrac{5}{2} x_3 = \tfrac{5}{2} \end{cases}$$

triangular system...
can solve by (back) substitution...

interpret this as a way of factoring matrices
This is one of the matrices in the factorization of A.

Solve it

only one variable here
solve for x_3 first

$$x_3 = -1$$

solve for x_2 next

$$x_2 = \frac{-9-15x_3}{2} = \frac{-9-15(-1)}{2} = \frac{6}{2} = 3$$

solve for x_1 last

$$x_1 = \frac{8+3x_3}{1} = 8 + 3(-1) = 5$$

The solution is

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}}$$

Left hand side

$$x_1 \quad -3x_3$$
$$2x_2 + 15x_3$$
$$-\tfrac{5}{2}x_3$$

$$U = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 15 \\ 0 & 0 & -5/2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix}$$

We will write

$A = LU$ where L is a lower triangular matrix yet to be found...

... matrix factorization of A