$[\, A \mid b \,]$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$\boxed{r_2 \leftarrow r_2 - 2r_1}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right]} \end{array}$$

eliminate

$\boxed{r_3 \leftarrow r_3 - \frac{1}{2} r_2}$

$5 - \frac{15}{2} = \frac{10-}{}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5/2 & 5/2 \end{array}\right]} \end{array}$$

to get from A to U I performed two elimination steps...

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 15 \\ 0 & 0 & -5/2 \end{bmatrix}$$

To get from U to A, then undo the elimination steps in reverse order...

$\boxed{r_2 \leftarrow r_2 - 2r_1}$

$\boxed{r_3 \leftarrow r_3 - \frac{1}{2} r_2}$

undo them in reverse order

$\boxed{r_3 \leftarrow r_3 + \frac{1}{2} r_2}$

$\boxed{r_2 \leftarrow r_2 + 2r_1}$

Matrix L represents these two steps...

Want factorization $A = LU$

To figure out what L is we need to think about the elementary row operations as linear functions...and consequently represent them using matrices...

$r_3 \leftarrow r_3 + \frac{1}{2} r_2$ is this a linear function?

row col.

3,2

Yes its input is a matrix... output another matrix....

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

first subscript is the row the second is the column...

row operation gives this

cols

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \text{rows} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+\frac{1}{2}a_{21} & a_{32}+\frac{1}{2}a_{22} & a_{33}+\frac{1}{2}a_{23} \end{bmatrix}$$

3,2 entry

In general the row operation

$r_i \leftarrow r_i + \alpha r_j$

is the same a matrix mult by a matrix with 1's on the diagonal, α in the (i,j) position and 0's everywhere else...

row operations

$$\begin{cases} ① & r_i \leftarrow r_i - \alpha r_j & i \neq j \\ ② & r_i \leftrightarrow r_j & i \neq j \\ ③ & r_i \leftarrow \alpha r_j & \alpha \neq 0 \end{cases}$$

The identity matrix is the matrix that corresponds to the identity function...

$f(x) = x$ ← identity function... it does nothing...

really in 3-dimension.

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ 0x_1 + x_2 + 0x_3 \\ 0x_1 + 0x_2 + x_3 \end{bmatrix}$$

Thus the matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

That is it's the matrix with 1's on the diagonal and 0's everywhere else...

$$r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 2 & 7 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 5 & 1 & 3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$\underbrace{r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

do the row operation on the identity matrix

The same ...

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

To find the matrix for the row operation, just perform the row operation on the identity matrix.

Check.....

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 7 \\ 5 & 1 & 3 \\ -2 & 3 & 0 \end{bmatrix}$$

$\boxed{r_3 \leftarrow r_3 + \frac{1}{2} r_2}$ $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$ This one first

$\boxed{r_2 \leftarrow r_2 + 2 r_1}$ $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then this one second

(2,1) position

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}}_{\text{matrix } L}$$

luck that the row operations don't interact when they are performed in the reverse order...

Therefore, $A = LU$ or specifically...

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 15 \\ 0 & 0 & -5/2 \end{bmatrix}$$

Check this at home...