To find the matrix for the row operation, just perform the row operation on the identity matrix.

Elementary row operations

all reversible

1. Elimination step $\quad r_i \leftarrow r_i - \alpha r_j, \quad i \neq j$
2. Row swap $\quad r_i \leftrightarrow r_j \quad i \neq j$
3. rescaling $\quad r_i \leftarrow \alpha r_i \quad \alpha \neq 0$

$A \in \mathbb{R}^{3\times3}$ where $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & -1 \\ 5 & 2 & 4 \end{bmatrix}$

Want to represent the rescaling operation $\underbrace{r_3 \leftarrow \tfrac{1}{2} r_3}_{\text{not very useful thing to do...}}$

as a matrix multiplication...

Since working with 3x3 matrices, use 3x3 identity

$$\underbrace{[r_3 \leftarrow \tfrac{1}{2} r_3]}_{\text{means the matrix corresponding to the operation}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$[r_3 \leftarrow \tfrac{1}{2} r_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

already

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1. (pivots...)
5. Each leading 1 is the only nonzero entry in its column.

extra conditions

pivots

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5/2 & 5/2 \end{bmatrix}$$

already in Echelon form from before.

Continue performing row operations to reduce the echelon form.

rescaling so 4 hold

$r_2 \leftarrow \frac{1}{2} r_2 \qquad r_3 \leftarrow \frac{-2}{5} r_3$

make zero

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

pivot
use it to make zeros in the other rows

More elimination so 5 holds

$r_1 \leftarrow r_1 + 3 r_3$

$r_2 \leftarrow r_2 - \frac{15}{2} r_3$

$-\frac{9}{2} - \frac{15}{2}(-1) = \frac{15-9}{2} = \frac{6}{2} = 3$

$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & \text{rhs} \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1
\end{array}\right]
$$

reduced row-echelon form...

linear function on the LHS is $\boxed{\text{identity func.}}$

4. The leading entry in each nonzero row is 1.

pivots...

5. Each leading 1 is the only nonzero entry in its column.

algebraic form...

$$
\begin{cases}
x_1 = 5 \\
x_2 = 3 \\
x_3 = -1
\end{cases}
\qquad \text{Solution} \qquad x = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}
$$

consistency. free variables

Existence and Uniqueness Questions

When solving $Ax = b$

① Is there a solution?: Yes means consistent

② How many solutions are there?

What could have happend?

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

still in reduced row-echelon form ...

linear function on the LHS is $\boxed{\text{identity func.}}$

Suppose the augmented matrix looks like this,

4. The leading entry in each nonzero row is 1. (pivots...)
5. Each leading 1 is the only nonzero entry in its column.

algebraic form

$$\begin{cases} x_1 + 2x_3 = 5 \\ x_2 + 6x_3 = 3 \\ x_4 = -1 \end{cases}$$

solve

View x_3 as a free parameter

$$x_1 = 5 - 2x_3$$
$$x_2 = 3 - 6x_3$$
$$x_3 = x_3$$
$$x_4 = -1$$

factor out x_3

$$x = \begin{bmatrix} 5 - 2x_3 \\ 3 - 6x_3 \\ x_3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_3$$

free variable

- For any value of x_3 you get another solution...
- Solutions are not unique...

Back to reduced row echelon form

pivot columns or variables...

free variable

$$\begin{array}{ccccc} P & P & F & P & \\ x_1 & x_2 & x_3 & x_4 & \text{rhs} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

pivots

Suppose instead the augmented matrix turned out like this

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{rhs} \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

added extra row

not in reduced row echelon form

$r_4 \leftarrow \frac{1}{6} r_4$

$r_1 \leftarrow r_1 - 5r_4$

$r_2 \leftarrow r_2 - 3r_4$

$r_3 \leftarrow r_3 + r_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

reduced...

looks like a pivot but oops it was supposed to be RHS..

causes problem

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

inconsistent no solution...