

To find the matrix for the row operation, just perform the row operation on the identity matrix.

Elementary row operations

all reversible

- ① Elimination step $r_i \leftarrow r_i - \alpha r_j, \quad i \neq j$
- ② Row swap $r_i \leftrightarrow r_j, \quad i \neq j$
- ③ Rescaling $r_i \leftarrow \alpha r_i, \quad \alpha \neq 0$

$A \in \mathbb{R}^{3 \times 3}$ where $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

Want to represent the rescaling operation $r_3 \leftarrow \frac{1}{2} r_3$ as a matrix multiplication...

not very useful thing to do...

Since working with 3×3 matrices, use 3×3 identity

$$\left[r_3 \leftarrow \frac{1}{2} r_3 \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

means the matrix corresponding to the operation

$$\left[r_3 \leftarrow \frac{1}{2} r_3 \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

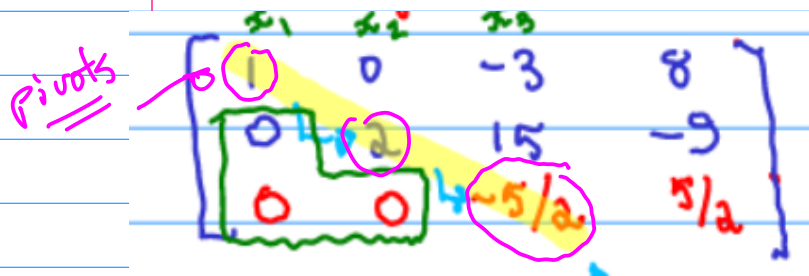
1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

already

extra conditions



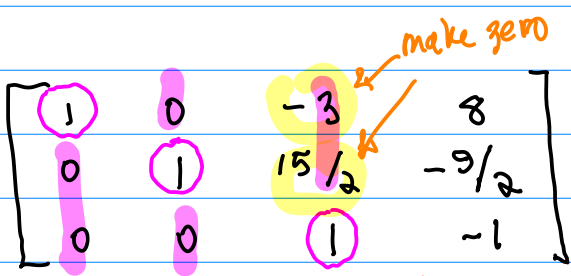
already in Echelon form from before.

Continue performing row operations to reduce the echelon form.

rescaling so 1 holds

$$r_2 \leftarrow \frac{1}{2} r_2$$

$$r_3 \leftarrow -\frac{2}{5} r_3$$



more elimination so 5 holds

$$r_1 \leftarrow r_1 + 3r_3$$

$$r_2 \leftarrow r_2 - \frac{15}{2} r_3$$

$$-\frac{9}{2} - \frac{15}{2}(-1) = \frac{15-9}{2} = \frac{6}{2} = 3$$

$$\begin{array}{ccc|c}
 x_1 & x_2 & x_3 & \text{rhs} \\
 \hline
 1 & 0 & 0 & 5 \\
 0 & 1 & 0 & 3 \\
 0 & 0 & 1 & -1
 \end{array}$$

reduced row-echelon form ...

linear function on the RHS is identity func.

- 4. The leading entry in each nonzero row is 1.
- 5. Each leading 1 is the only nonzero entry in its column.

algebraic form ...

$$\begin{cases}
 x_1 = 5 \\
 x_2 = 3 \\
 x_3 = -1
 \end{cases}
 \quad \text{Solution} \quad x = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

consistency. free variables
Existence and Uniqueness Questions

When solving $Ax = b$

- ① Is there a solution? Yes means consistent
- ② How many solutions are there?

what could have happened?

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \text{rhs} \\ 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

still in reduced row-echelon form ...

Linear function on the RHS is identity func.

Suppose the augmented matrix looks like this,

4. The leading entry in each nonzero row is 1.
 pivots...
5. Each leading 1 is the only nonzero entry in its column.

Algebraic form

$$\begin{cases} x_1 + 2x_3 = 5 \\ x_2 + 6x_3 = 3 \\ x_4 = -1 \end{cases}$$

Solve

View x_3 as a free parameter

$$\begin{aligned} x_1 &= 5 - 2x_3 \\ x_2 &= 3 - 6x_3 \\ x_3 &= x_3 \\ x_4 &= -1 \end{aligned}$$

$$x = \begin{bmatrix} 5 - 2x_3 \\ 3 - 6x_3 \\ x_3 \\ -1 \end{bmatrix}$$

factor out x_3

$$= \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_3$$

free variable

- For any value of x_3 , you get another solution...
- Solutions are not unique...

Back to reduced row echelon form

pivot columns or variables...

Free variable

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \text{rhs} \\ 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 6 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

pivots

Suppose instead
the augmented
matrix turned out like this

$$\begin{array}{c}
 x_1 \quad x_2 \quad x_3 \quad x_4 \quad \text{rhs} \\
 \left[\begin{array}{cccc|c}
 1 & 0 & 2 & 0 & 5 \\
 0 & 1 & 6 & 0 & 3 \\
 0 & 0 & 0 & 1 & -1 \\
 0 & 0 & 0 & 0 & 6
 \end{array} \right]
 \end{array}$$

added
extra row
not in reduced
row echelon
form

$$r_4 \leftarrow \frac{1}{6} r_4$$

$$r_1 \leftarrow r_1 - 5r_4$$

$$r_2 \leftarrow r_2 - 3r_4$$

$$r_3 \leftarrow r_3 + r_4$$

$$\begin{array}{c}
 \left[\begin{array}{cccc|c}
 1 & 0 & 2 & 0 & 0 \\
 0 & 1 & 6 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1
 \end{array} \right]
 \end{array}$$

reduced...

looks like a pivot
but oops it was
supposed to be RHS.

causes problem

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

inconsistent
no
solution...