From 1.5 #19

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

Homogeneous version of #19

5.
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

Solve this,,, first do 5. Create Echelon form of A where

pivot

$$A = \begin{bmatrix} \boxed{1} & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

eliminate the -4 to zero

$r_2 \leftarrow r_2 + 4r_1$

(2,1)

row col

pivot

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & \boxed{3} & 6 \\ 0 & -3 & -6 \end{bmatrix}$$

eliminate the -3

$r_3 \leftarrow r_3 + 1r_2$

(3,2)

row column

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form of A

$= U$ upper triangular

Write $A = LU$

Lower triangular with 1's on diagonal

Recall, L contains the multipliers from the elimination steps...

Swap the sign of 4 to -4 because L undoes what the row operations did.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The 1 changes to -1

$A = LU$

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Note we will solve $Ax = 0$
using $LUx = 0$

Thus; writing $y = Ux$ we have

$$\begin{cases} Ly = 0 \\ Ux = y \end{cases}$$ a system of systems...

for homogenous problems, since L has 1's on the diagonal, the only solution y such that $Ly = 0$ is $y = 0$. Thus solve $Ux = 0$.

note, for homogenous problem, I can take a shortcut and just solve $Ux = 0$

For example

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

means

$$\begin{cases} y_1 = 0 \\ -4y_1 + y_2 = 0 \\ -1y_2 + y_3 = 0 \end{cases}$$

thus $y_2 = 0$

thus $y_3 = 0$

so $y = 0$

Solve $Ux = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ P & P & F & \end{array}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solve for the pivot variables in term of the free vbls.

$$x_1 + 3x_2 + x_3 = 0$$

$$3x_2 + 6x_3 = 0$$

by back substitution

next solve for x_1

$$\boxed{x_2 = -2x_3}$$

$$x_1 = -3x_2 - x_3 = -3(-2x_3) - x_3 = \boxed{5x_3}$$

Note always that free variables are equal to themselves, so

$$x_3 = x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3}$$

final answer for Question 5.

Back to question 19: Now we solve $Ax = b$

where $b = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

Solve

$$LUx = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

Thus $\begin{cases} Ly = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \\ \\ Ux = y \end{cases}$ ← solve this system of systems...

Now $y \neq 0$ so x is different than last time, But L and U are the same...

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

$$\left.\begin{aligned} y_1 &= 1 \\ -4y_1 + y_2 &= -1 \\ -1y_2 + y_3 &= -3 \end{aligned}\right\}$$ so substitute to obtain

$y_1 = 1$

$y_2 = -1 + 4y_1 = -1 + 4 = 3$

$y_3 = -3 + y_2 = -3 + 3 = 0$

Solve $Ux = y$ where $y = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$$\begin{array}{ccc} P & P & F \\ x_1 & x_2 & x_3 \end{array}$$

backwards

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$ ← This last line says $0 = 0$ which is OK!

Solve by back substitution, solve for pivots in terms of the free variables

$$x_1 + 3x_2 + x_3 = 1$$
$$3x_2 + 6x_3 = 3$$

$$x_2 = \underbrace{\frac{3 - 6x_3}{3}} = 1 - 2x_3$$

$$x_1 = 1 - 3x_2 - x_3 = 1 - 3(1 - 2x_3) - x_3$$
$$= -2 + 5x_3$$

$$x_3 = x_3$$

Answer

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3$$

Note this part is the soln to the homogeneous equation.