Two way of viewing Ax

① As dot product (row way)

row

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 0 \\
-4x_1 - 9x_2 + 2x_3 &= -0 \\
-3x_2 - 6x_3 &= -0
\end{aligned}
$$

Dot product:

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 3x_2 + x_3$$

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,3,1)\cdot(x_1,x_2,x_3) \\ (-4,-9,2)\cdot(x_1,x_2,x_3) \\ (0,-3,-6)\cdot(x_1,x_2,x_3) \end{bmatrix}$$

The first we we wrote

dot products between rows of the first and columns of the 2nd

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

answer

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

uses rows of the matrix ...

a different way of looking at the same product: Column way...

$$
\begin{aligned}
1x_1 + 3x_2 + 1x_3 &= 0\\
-4x_1 - 9x_2 + 2x_3 &= 0\\
0x_1 - 3x_2 - 6x_3 &= 0
\end{aligned}
$$

columns in the matrix, factoring out the x's.

$$
Ax = \underbrace{\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ -9 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} x_3}_{\text{written the matrix product in terms of columns...}}
$$

row characterization — column characterization

$$
\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,3,1)\cdot(x_1,x_2,x_3) \\ (-4,-9,2)\cdot(x_1,x_2,x_3) \\ (0,-3,-6)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ -9 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} x_3
$$

Chapter 1.7: Linear independence.

$$
x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

this vector equation

$$
A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}
$$

then $Ax = 0$

Definition

Same thing as asking whether $Ax=0$ has a unique solution or many...

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

column repres. of $Ax=0$

the solution $x=0$

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

if the trivial solution is the only one that means the solution is unique.

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

column repres. of $Ac=0$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

means there are solutions other than the trivial solution...

Problem 5

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

Answer:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3}$$

lots of solutions because x_3 is a free variable and can be chosen to be anything.

Independent	Dependent
unique solution	many solutions to $Ax=0$
No free variables	Yes free variable
Every column has a pivot.	Some columns don't have pivots

all these things mean the same

all these things mean the same

- Figure out the pivots and free variable by making the Echelon form using Gaussian elimination.

Characterization of Linearly Dependent Sets

case where there are free variables...

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

echelon form of A

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \approx U$$

How did we find U from A? Row operations (elimination steps)

$Ax = 0$ meant the same as $Ux = 0$

This means if there is a dependency relation between the columns of A, those same dependency relations hold for the columns of U.

Note the L is always invertible (because row operations are reversible) and this means the columns of L are independent.

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \approx U$$

last column is a combination of the first two...

$$\begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The last vector is a linear combination of the first two...

Recall also the answer to $Ax = 0$ was

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3}$$

How to see those are the same 5 and 2? Set $x_3 = 1$.

$x = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $Ax = 0$