Dear class, I looks like I deleted Monday's lecture notes by accident so I can not post an exact transcript of what appeared in class.

Instead I'm posting this different version of the same thing.

Of the geometric visualizations of linear functions in Section 1.9 note that most of them are given by elementary row operations. These are

1. $r_i \leftarrow r_i - \alpha r_j$ elimination steps shearing
2. $r_i \leftrightarrow r_j$ row swaps reflections
3. $r_i \leftarrow \alpha r_j$ scaling operations stretch or shrink

Notably, however, are the rotations which aren't given by one of the row operations. Before discussing these, we ask ...

Question: Given a linear function $f(x)$ how can one find the matrix A such that $f(x) = Ax$ by only testing a few inputs of f and checking the outputs?

Usually we think of $f(x)$ as the left side of a system of linear equations. For example, given

$$2x_1 + 3x_2 = 5$$
$$x_1 - 2x_2 = 7$$

the function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

represents the left side. In this case the matrix A such that $f(x) = Ax$ can just be read off as the coefficients of the x_i's. Namely

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}.$$

What is special about f is that it's linear. Any function that can be represented by a matrix is linear.

What does linear mean?

A function f mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is linear if

① $\underbrace{f(x+y)}_{\text{add inputs}} = \underbrace{f(x) + f(y)}_{\text{add outputs}}$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$

② $\underbrace{f(\alpha x)}_{\text{rescale input}} = \underbrace{\alpha f(x)}_{\text{rescale output}}$ for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$

Note that both these properties state that some operation on the inputs is the same as an operation on the outputs.

So, how can these two properties be used to figure out what the matrix A is?

We begin by defining the standard basis e_k to be the vectors with the k'th entry given by 1 and the others 0.

Thus if $n=4$ then the standard basis of $\mathbb{R}^4$ is

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{1st entry}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{2nd entry}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{3rd entry} \quad \text{and} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{4th entry}$$

These vectors are also called the unit vectors along the coordinate axis, maybe labeled $\hat{i}, \hat{j}$ and $\hat{k}$ in vector calculus.

Now given any $x \in \mathbb{R}^4$ we may write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$$

Thus, any x can be written as a linear combination of the standard basis vectors e_k.

Now return to the question:

Question: Given a linear function $f(x)$ how can one find the matrix A such that $f(x) = Ax$ by only testing a few inputs of f and checking the outputs?

Suppose we know the values of f when the e_k's are used for inputs. In particular, let

$$v_1 = f(e_1), \quad v_2 = f(e_2), \quad \ldots, \quad v_n = f(e_n)$$

Note that since $f: \mathbb{R}^n \to \mathbb{R}^m$ then the output $v_k \in \mathbb{R}^m$.

Knowing what the v_k's are is enough to find A. In fact the v_k's are the columns of A.

$$A = \left[\begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_n \end{array}\right] \in \mathbb{R}^{m \times n}$$

How? Why? The properties

① $f(x+y) = f(x) + f(y)$

② $f(\alpha x) = \alpha f(x)$

allows us to figure out $f(x)$ by just knowing $f(e_1), f(e_2), \ldots, f(e_n)$.

For simplicity we suppose $n=4$ and $m=3$. Then we find

$$v_1 = f(e_1) \;,\; v_2 = f(e_2) \;,\; v_3 = f(e_3) \text{ and } v_4 = f(e_4)$$

For example, it might happen that

$$f\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix},\; f\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\-1\end{bmatrix},\; f\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\5\end{bmatrix} \text{ and } f\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\3\\1\end{bmatrix}$$

Now consider any $x \in \mathbb{R}^4$. From before

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$$

Therefore

$f(x+y) = f(x) + f(y)$

$$f(x) = f(x_1 e_1) + f(x_2 e_2) + f(x_3 e_3) + f(x_4 e_4)$$

properties of linear functions

$$= x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3) + x_4 f(e_4)$$

$f(\alpha x) = \alpha f(x)$

$$= \underbrace{x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4}_{\text{this is the column representation of the matrix product}}$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} x_1\\x_2\\x_3\\x_4 \end{bmatrix} = Ax \quad \text{where} \quad A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

Recalling that

$$f\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix},\; f\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\-1\end{bmatrix},\; f\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\5\end{bmatrix} \text{ and } f\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\3\\1\end{bmatrix}$$

yields

$$A = \begin{bmatrix} v_1 | v_2 | v_3 | v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & -1 & 5 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$$

rows, columns

matrices with entries given by real numbers.

Now, back to Section 1.9 and rotations:

SOLUTION $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $v_1 = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $v_2 = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$. See Figure 1. By Theorem 10,

$$A = \begin{bmatrix} v_1 | v_2 \end{bmatrix} = A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

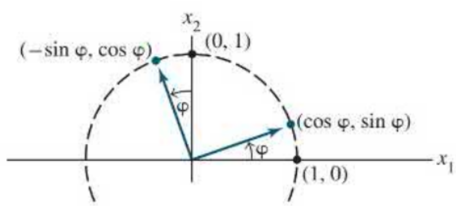

FIGURE 1 A rotation transformation.

Thus, what's actually going on is figuring out the matrix of an unknown linear function "the rotation" by plugging in e_1 and e_2.

Begin Chapter 2.

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$ $\quad (f+g)(x) = (g+f)(x)$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Remember that the matrices stand for linear functions. Here addition of matrices means addition of the functions.

Recall, the addition of functions is defined as

$$\underbrace{(f+g)(x)}_{\text{add functions}} = \underbrace{f(x)+g(x)}_{\text{add their outputs}}$$

In our case the outputs are just vectors. Since vectors can be added in any order, for example,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The we have

$$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$$

Note, the domains and ranges of both functions need to be the same for the above to make sense.

Thus

same n

$$f: \mathbb{R}^n \to \mathbb{R}^m \qquad g: \mathbb{R}^n \to \mathbb{R}^m$$

same m

which means the matrices

same

$$A \in \mathbb{R}^{m \times n} \quad \text{and} \quad B \in \mathbb{R}^{m \times n}$$

same

must have the same dimensions.

We'll discuss exactly how to add $A+B$ next time.