If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—

$f(x) = Ax$ $\qquad$ $f:$ Domain $\rightarrow$ Range

$m = 3$ $\qquad$ $n = 4$ $\qquad$ $f: \mathbb{R}^4 \rightarrow \mathbb{R}^3$

rows (output) $\qquad$ cols (input)

$A \in \mathbb{R}^{3 \times 4}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$ so 4 in this column vector.

4 cols

$x \in \mathbb{R}^4$

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 6 & 3 \\ 3 & 5 & -1 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

answer $f(x) = Ax \in \mathbb{R}^3$

3 vector

Shortcut to check the dimensions:

$$\underset{3\times\cancel{4}}{A}\ \underset{\cancel{4}}{x} = \underset{3}{y}$$

← write the dimensions underneath.

both are 4's so this mutiplication is okay...

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

$f(x) = Ax$ $\quad A \in \mathbb{R}^{m \times n}$ $\quad f: \mathbb{R}^n \to \mathbb{R}^m$

$g(x) = Bx$ $\quad B \in \mathbb{R}^{m \times n}$ $\quad g: \mathbb{R}^n \to \mathbb{R}^m$

why the same dimension as A?

$A+B$ means add the functions $f+g$

add the outputs together. so the output vector has the same length

$$(f+g)(x) = f(x) + g(x)$$

same input is used in both f and g...

How to add the matrices?

$A+B$ should be the linear function that corresponds to $(f+g)(x)$

$$(f+g)(x) = (A+B)x$$ definition of $A+B$

meaning of addition of funtions

$$\|$$

$$f(x)+g(x) = Ax + Bx$$

Therefore $(A+B)x = Ax+Bx$ ...

Example: $A \in \mathbb{R}^{3\times4}$, $B \in \mathbb{R}^{3\times4}$

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 6 & 3 \\ 3 & 5 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 & 1 & 1 \\ 1 & 4 & -3 & 2 \\ 6 & -2 & 4 & 3 \end{bmatrix}$$

Nice thing: $A+B$ is easy ... add the matrice just like vectors... that is element by element..

$$A+B = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 6 & 3 \\ 3 & 5 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 1 & 1 \\ 1 & 4 & -3 & 2 \\ 6 & -2 & 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 3 & 2 \\ 3 & 4 & 3 & 5 \\ 9 & 3 & 7 & 3 \end{bmatrix}$$

Back to matrix multiplication...

column representation

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

what does this mean?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & -2 & 3 \end{bmatrix}$$

$$\underbrace{2\times 2 \qquad\qquad 2\times\square}_{\text{need same}}$$

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & -2 & 3 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 2 & -3 & 5 \\ 11 & 4 & -5 & 9 \end{bmatrix} \end{array}$$

← answer for AB

$$\begin{array}{cc} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 \\ 11 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{array}$$

The motivation for block matrices...

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad B = \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 2 & 1 & -2 & 3 \end{array}\right] = [\,C \mid D\,]$

$C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$$AB = [A][\,C \mid D\,] = [\,AC \mid AD\,]$$

Multiply this out

$$\begin{array}{cc} & [\,C \mid D\,] \\ [A] & [\,AC \mid AD\,] \end{array}$$

answer

ROW–COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

row col.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

the entries in the product AB are just dot products between rows of A and columns of B.

Goal, in part, is to build reading for this kind of notation...

a. $A(BC) = (AB)C$ (associative law of multiplication)

$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ why?

meaning

$$(f \circ (g \circ h))(x) = f\big((g \circ h)(x)\big) = f\big(g(h(x))\big)$$

same thing...

$$((f \circ g) \circ h)(x) = (f \circ g)\big(h(x)\big) = f\big(g(h(x))\big)$$