If a function f has the same domain as its range then you can compose it with itself:

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \quad \text{(linear function)}$$

If $m = n$ then $f \circ f(x) = f(f(x))$ makes sense.

If f is linear, then $f(x) = Ax$ for some matrix

If $m = n$ then A is a square matrix

$$A \in \mathbb{R}^{n \times n}$$

rows (output) cols (input)

and it makes sense to write

$$\underset{n \times \cancel{n} \; \cancel{n} \times n}{A^2 = AA} \quad \in \mathbb{R}^{n \times n}$$

$$A^3 = AAA$$

$$A^k = \underbrace{A \dots A}_{k \text{ times}} \quad \text{means} \quad (\underbrace{f \circ \dots \circ f}_{\text{compose } f \text{ with itself } k \text{ times}})(x)$$

Review of powers from Calculus

$$a^k = \underbrace{a \cdot a \dots a}_{k \text{ times}}$$

number

algebra $\left\{ \; a^{-1} = \dfrac{1}{a} \right.$

why so that $a^x a^y = a^{x+y}$

$a^{-1} a^1 = a^0 = 1$ and it really is

$$a^{-k} = \frac{1}{a^k}$$

Polynomial so it's still $\underbrace{\text{algebra}}$

$a^{1/2} = \sqrt{a}$ or the solution to $x^2 - a = 0$

Note there are two solutions..

In other words $a^{1/2}$ is any number such that

$$(a^{1/2})(a^{1/2}) = a.$$

$a^{p/q} = \sqrt[q]{a^p}$

Thus

$$\underbrace{(a^{p/q}) \cdots (a^{p/q})}_{q \text{ times}} = \underbrace{a \cdot \cdots a}_{p \text{ times.}}$$

has meaning using algebra.

$a^{\sqrt{2}}$ problem: $\sqrt{2} \neq \frac{p}{q}$ for any p and q

because it's irrational...

a^{π} And π is transcendental...

but what does it mean in the exponent...

We know what $\sqrt{2}$ is.

In calculus we write $a^{\sqrt{2}} = \lim_{\frac{p}{q} \to \sqrt{2}} a^{p/q}$

Calculus idea: define something using the limit of approximations...

Things fit together and we find

$$a^{\sqrt{2}} = \exp\left((\ln a)\sqrt{2}\right)$$

$\ln a = \int_1^a \frac{1}{t}\,dt$ and exp is the inv of ln.

Back to linear algebra..

A^{-1} is the matrix such that $A^{-1}A = I$

Suppose $f(x) = Ax$ ... what is the matrix for $f^{-1}(y)$?

If f is a linear function is it obvious that the inverse function is also linear (provided f^{-1} exists) invertible.

- Intuitively: f is a line and if you swap the x and y axis of the graph to find f^{-1} its still a line.

Algebraic reason: Let f be a linear function.

$f(x) = y$ $\qquad$ $f(w) = z$ $\qquad$ so $\begin{cases} x = f^{-1}(y) \\ w = f^{-1}(z) \end{cases}$

$f(x) + f(w) = f(x+w)$ since f is linear

Since f^{-1} is assumed to exist, apply it to both sides.

$$f^{-1}(f(x) + f(w)) = f^{-1}(f(x+w)) = x+w$$

$$f^{-1}(y+z) = x+w$$

$$\boxed{f^{-1}(y+z) = f^{-1}(y) + f^{-1}(z)}$$

algebraic property for f^{-1} to be linear...

What's the other property needed to show f^{-1} is linear?

$$\boxed{f^{-1}(\alpha y) = \alpha f^{-1}(y)}$$ ← also need this

over the weekend think about this

Finding the matrix A^{-1} that represents f^{-1} is possible because f^{-1} is linear and we do it by plugging $e_1, e_2, \ldots$ and so on...

$$v_1 = f^{-1}(e_1), \quad v_2 = f^{-1}(e_2) \cdots \quad v_n = f^{-1}(e_n)$$

Then

$$A^{-1} = \begin{bmatrix} v_1 \,\big|\, v_2 \,\big|\, \cdots \,\big|\, v_n \end{bmatrix}$$

So what are the v_k's and how to find them.

$$v_1 = f^{-1}(e_1), \quad v_2 = f^{-1}(e_2) \cdots \quad v_n = f^{-1}(e_n)$$

means

$$f(v_1) = e_1, \quad f(v_2) = e_2, \ldots, f(v_n) = e_n$$

or

$$\underbrace{Av_1 = e_1 \qquad Av_2 = e_2 \qquad Av_n = e_n}$$

these are systems of linear equations that can be solved...

We do this in section 2.2

But what about $A^{1/2}$?

How to find a matrix such that

$$(A^{1/2})(A^{1/2}) = A \quad ?$$

To find $A^{1/2}$ we first solve the eigenvalue – eigenvector problem for A.
Chapter 5 (I think) ...

Note the eigenvalue-eigenvector problem basically converts matrix multiplication into scalar multiplication.
After that all the Calculus stuff to find things like $A^{1/2}$, $A^{\sqrt{2}}$ and more work.

In fact we end up with $\ln A$, $\exp(A)$ and $\sin A$, $\cos A$ and sorts of things where A is a matrix.

The last part of 2.) is about transposes...

transpose: switch rows with columns in a matrix.

$$Ax = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

dot the rows of the first matrix into the columns of the second...

If I want to write the first matrix second and the second first, yet still have the same dot products I need transpose the matrices...

$$x^T A^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -4 \end{bmatrix}$$

This is the transpose of the first answer

In symbols:

$$(Ax)^T = x^T A^T$$

One last thing

$$Ax \cdot y = (Ax)^T y = x^T A^T y = x \cdot A^T y$$

$$Ax \cdot y = x \cdot A^T y$$

Thus the transpose on A is what happens when the matrix 'jumps over the dot' in the dot product.