ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

41. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ **42.** $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

Solving: $Av_1 = e_1$, $Av_2 = e_2$ and $Av_3 = e_3$

#42

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} r_2 \leftarrow r_2 - 4r_1 \\ r_3 \leftarrow r_3 + 2r_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 - 2r_1$$

not a pivot in every row

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

non-zero on the right

pivot is missing — no inverse...

zero on the left

another example.

#41

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 - 2r_1 \end{array}$$

sign changes

undo these row operations

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & \boxed{1} & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \qquad r_3 \leftarrow r_3 + 3r_2$$

RHS

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & \boxed{1} & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

echelon form...

aside $A = LU$

$$U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

then could use LU factorization to solve $Av_1 = e_1$, $Av_2 = e_2$ and $Av_3 = e_3$

Alternatively find reduced echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & \boxed{2} & 7 & 3 & 1 \end{array}\right]$$

$r_2 \leftarrow r_2 + r_3$

$r_1 \leftarrow r_1 + r_3$

don't change

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & \boxed{2} & 7 & 3 & 1 \end{array}\right]$$

rescale so all pivots are 1.

$r_3 \leftarrow \frac{1}{2} r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

this is A^{-1}

reduced echelon form

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

Thus,

$$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = I$$

laws of exponents $A^{1-1} = A^0 = I$

note A^0 doesn't change the exponent when multiply by it... the only matrix that doesn't change anything is the identity, so $A^0 = I$

$$A^2 A^{-1} = A^{2-1} = A^1 = A$$

$$AA^{-2} = A^{1-2} = A^{-1} \qquad \text{where } A^{-2} = A^{-1}A^{-1}$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix. reduced echelon form of A.

c. A has n pivot positions. pivot in every row means no row is zero.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. free variable so less pivots... not enough pivots

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.