The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix. reduced echelon form of A.

c. A has n pivot positions. pivot in every row means no row is zero.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. free variable so less pivots... not enough pivots

Same

e. The columns of A form a linearly independent set. if not, there are free variables

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. the only solution to $Ax=0$ is the trivial solution

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$. ✓

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. ✓

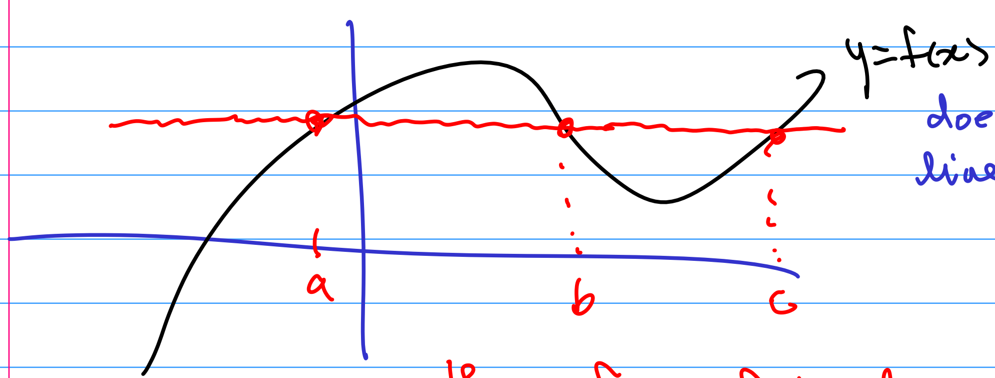
j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix. talk about later...

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one does this mean it's invertible...

the horizontal line test for invertibility



doesn't pass horizontal line test

then $f(a) = f(b) = f(c)$ but a, b and c are different...

Clear if f is not one-to-one then it's not invertible...

Condition of $f:\mathbb{R}^n \to \mathbb{R}^n$ being onto

Note that not only does f have to be one-to-one but the range of f has to be all of $\mathbb{R}^n$, so any $y \in \mathbb{R}^n$ can be seen as the output

$$y = f(x) \text{ for some } x \in \mathbb{R}^n.$$

If one-to-one that means for every $x \neq w$ that $f(x) \neq f(y)$.

$$Ax \neq Aw \text{ for every } x \neq w \ldots$$

$$Ax - Aw \neq 0 \text{ for every } x \neq w$$

linearity

$$A(x-w) \neq 0 \text{ for every } x \neq w$$

Thus the only solution to $Ax = 0$ is $x = 0$.

because anything not equal zero can be written as $x-w$ where $x \neq w$.

I.e. the only time $A(x-w) = 0$ is when $x = w$.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. the only

This says that $f:\mathbb{R}^n \to \mathbb{R}^n$ is onto $\mathbb{R}^n$.

Thus we can solve $Av_1 = e_1, \ldots, Av_n = e_n$

$$A^{-1} = \left[v_1 \middle| v_2 \middle| \cdots \middle| v_n \right]$$

It follows that

column representation of matrix mult...

$$AA^{-1} = A\left[v_1 | v_2 | \cdots | v_n\right] = \left[Av_1 | Av_2 | \cdots | Av_n\right]$$

$$= \left[e_1 | e_2 | \cdots | e_n\right] = I$$

make this...

If $AA^{-1} = I$ does it follow that $A^{-1}A = I$?

In general $AB \neq BA$. Also if by chance

$AB = I$ it's not necessary that $BA = I$.

However, if $A \in \mathbb{R}^{n \times n}$ then it's true that $BA = I$ also...

Given...

$$AA^{-1} = I$$

$$AA^{-1}A = IA = A$$

$$A(A^{-1}A) = A$$

trying to show (understand why) $A^{-1}A = I$

need to know A is one to one... to conclude this... I only know A is onto...

① If A is onto then you can solve $Ax = b$ for every b.

② this means there is a pivot in each row of A.

③ If there is a pivot in every row and A is square this means there is a pivot in every column.

④ If there is a pivot in every column there are no free variables.

⑤ That means A is one-to-one..

Chapter 2.5

We have examples of finding LU already... but not very many of using it to solve $Ax=b$.

Solve $Ax=b$ where $A=$

$$\textbf{1. } A=\begin{bmatrix}3&-7&-2\\-3&5&1\\6&-4&0\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}-7\\5\\2\end{bmatrix}$$

$$A=\begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}\begin{bmatrix}3&-7&-2\\0&-2&-1\\0&0&-1\end{bmatrix}$$

L U

$Ax=b$

$LUx=b$

$y=Ux$

$\begin{cases} Ly=b \\ Ux=y \end{cases}$ solve this system of systems..

First solve $Ly=b$:

$$\begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}-7\\5\\2\end{bmatrix}$$

By substitution

$y_1=-7$

$-y_1+y_2=5$

$2y_1-5y_2+y_3=2$

$y_1=-7$

$y_2=5+y_1=12$ -2

$y_3=2-2y_1+5y_2$

$y_3 = 2 + 14 - 10 = 6$

makes me scared...

OK

$$y = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$

Next solve $Ux = y$ by substitution...