$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U \in \mathbb{R}^{3\times 4}$$ ← echelon form

what is L?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$$

How to solve a problem with this factorization

$LUx = b$ $\qquad \begin{cases} Ly = b \\ Ux = y \end{cases}$, $\quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Nothing exciting happens with $Ly = b$ — because L is invertible

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$y_1 = 1$

$3y_1 + y_2 = 2 \qquad y_2 = 2 - 3 = -1$

$-\frac{1}{2}y_1 - 2y_2 + y_3 = 3$

$y_3 = 3 + \frac{1}{2}y_1 + 2y_2$

$= 3 + \frac{1}{2} - 2 = 2\frac{1}{2} = \frac{3}{2}$

finished solving $Ly=b$ →

$$y = \begin{bmatrix} 1 \\ -1 \\ 3/2 \end{bmatrix}$$

Next solve $Ux = y$

$$\begin{array}{cccc} P & P & F & P \end{array}$$
$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3/2 \end{bmatrix}$$

Solve this by substitution... or make augmented matrix

$$\begin{bmatrix} 2 & -4 & 4 & -2 & 1 \\ 0 & 3 & -5 & 3 & -1 \\ 0 & 0 & 0 & 5 & 3/2 \end{bmatrix}$$

Make reduced echelon form to solve for x

Solve this How? by substitution (what we just did) or using the augmented matrix — other way

$2x_1 - 4x_2 + 4x_3 - 2x_4 = 1$

$3x_2 - 5x_3 + 3x_4 = -1$ — free, don't solve for x_3

$5x_4 = 3/2$

$x_4 = 3/10$

$$x_2 = \frac{-1 + 5x_3 - 3x_4}{3} = \frac{-1.9 + 5x_3}{3} = \frac{-1.9}{3} + \frac{5}{3}x_3$$

$$x_1 = \frac{1 + 4x_2 - 4x_3 + 2x_4}{2} = \frac{1 + \frac{4}{3}(-1.9 + 5x_3) - 4x_3 + 3/5}{2}$$

$$= \frac{\frac{-14}{15} + \frac{8}{3}x_3}{2} = -\frac{7}{15} + \frac{4}{3}x_3$$

$1 + \frac{4}{3}(-1.9) + \frac{3}{5}$

$= 1 - \frac{\cancel{4}^{2}}{3}\cdot\frac{19}{\cancel{10}_{5}} + \frac{3}{5}$

$= \frac{15}{15} - \frac{38}{15} + \frac{9}{15} = \frac{-14}{15}$

15 + 9 = 24; 38 − 24 = 14

$\frac{4}{3}\cdot 5 - 4 = \frac{20}{3} - \frac{12}{3} = \frac{8}{3}$

vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7/15 + 4/3\, x_3 \\ -19/30 + 5/3\, x_3 \\ x_3 \\ 3/10 \end{bmatrix} = \begin{bmatrix} -7/15 \\ -19/30 \\ 0 \\ 3/10 \end{bmatrix} + \begin{bmatrix} 4/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix} x_3$$

↖ free variable...

alternative method with augmented matrix

$$\begin{array}{cccc} P & P & F & P \end{array}$$
$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3/2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & -4 & 4 & -2 & 1 \\ 0 & 3 & -5 & 3 & -1 \\ 0 & 0 & 0 & 5 & 3/2. \end{array}\right]$$

now do row operations.., to create reduced row echelon form...

What is this?

$\text{Span}\{v_1, v_2, \dots, v_n\} = ?$ Example 3 vectors in $\mathbb{R}^4$.

$$\text{Span}\{v_1, v_2, v_3\} = \{v_1c_1 + v_2c_2 + v_3c_3 : c_1, c_2, c_3 \in \mathbb{R}\}$$

$$\{Ax : x \in \mathbb{R}^3\} \quad \text{where } A = \begin{bmatrix} v_1 | v_2 | v_3 \end{bmatrix}$$

Is it possible to get every vector $b \in \mathbb{R}^4$ by chosing a suitable vector $x \in \mathbb{R}^3$ such that $b = Ax$.

Thus $\text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^4$ means you can solve $Ax = b$ for every $b \in \mathbb{R}^4$.

rows columns

Since $A = \begin{bmatrix} v_1 | v_2 | v_3 \end{bmatrix} \in \mathbb{R}^{4 \times 3}$

has more rows than columns the echelon of A couldn't have a pivot in every row... because you can't have more pivots than columns and there are only 3 columns...

Thus there is a row of zeros which means $Ax = b$ can't be solved for all b.

and so

$$\text{span}\{v_1, v_2, v_3\} \neq \mathbb{R}^4$$

Solve $Ax=0$ for $A=\begin{bmatrix}1&3&4\\2&6&5\end{bmatrix}$

Use row operations... to find the reduced echelon form...

pivot

$$\begin{bmatrix}1&3&4\\2&6&5\end{bmatrix} \qquad r_2 \leftarrow r_2 - 2r_1$$

eliminate the 2

LU factorization

$$U=\begin{bmatrix}1&3&4\\0&0&-3\end{bmatrix}$$

$$L=\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

eliminate the 4

echelon for.

$$U=\begin{bmatrix}1&3&4\\0&0&-3\end{bmatrix} \text{ pivot} \qquad r_1 \leftarrow r_1 + \frac{4}{3} r_2$$

$$\begin{bmatrix}1&3&0\\0&0&-3\end{bmatrix} \qquad r_2 \leftarrow \frac{1}{-3} r_2$$

rescale this pivot

reduced row echelon form

$$\begin{array}{ccc} P & F & P \end{array}$$

$$\begin{bmatrix}1&3&0\\0&0&1\end{bmatrix}$$

Solving homogeneous equation (zero in the right side)

$$\begin{cases} x_1 + 3x_2 = 0 \\ x_3 = 0 \end{cases}$$

& Now solve for x... note x_2 is a free variable

solve

$$\begin{cases} x_3 = 0 \\ x_1 = -3x_2 \end{cases}$$

in vector form

Answer:

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3x_2\\x_2\\0\end{bmatrix}=\begin{bmatrix}-3\\1\\0\end{bmatrix}x_2$$

One more LU factorization problem

find the LU factorization of A

$$A=\begin{bmatrix}2&-6&6\\-4&5&-7\\3&5&-1\\-6&4&-8\\8&-3&9\end{bmatrix}$$

make all these zero

$r_2 \leftarrow r_2 + 2r_1$ ✓

$r_3 \leftarrow r_3 - \frac{3}{2} r_1$ ✓

$r_4 \leftarrow r_4 + 3r_1$ ✓

$r_5 \leftarrow r_5 - 4r_1$ ✓

$$\begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix}$$

zero here

$r_3 \leftarrow r_3 + 2r_2$

$r_4 \leftarrow r_4 - 2r_2$

$r_5 \leftarrow r_5 + 3r_2$

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ -4 & -3 & 0 & 0 & 1 \end{bmatrix}$$