A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

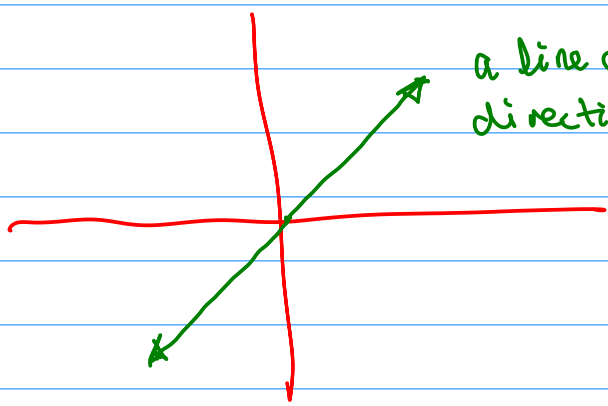
a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

related to definition of a linear function

One dimensional subspace of $\mathbb{R}^2$



a line extending to ∞ in both directions passing through the origin

Column Space and Null Space of a Matrix

- The column space is all values of b such that $Ax=b$ has a solution. This is the range of the function $f(x)=Ax$.

$$Col(A) = \{Ax : x \in \mathbb{R}^n\}$$

all possible outputs given any vector x input.

How to find Col(A)? compute it...

Is this really a subspace?

Definition of a linear function:

- $f(x+y) = f(x) + f(y)$ $\quad x, y \in \mathbb{R}^n$
- $f(\alpha x) = \alpha f(x)$ $\quad \alpha \in \mathbb{R}$

If $u, v \in \mathrm{Col}(A)$ why is $u+v$ also there?

$$\boxed{\mathrm{Col}\,A = \{ Ax : x \in \mathbb{R}^n \}.}$$

If $u \in \mathrm{Col}\,A$ there is an x so that $u = Ax$

$v \in \mathrm{Col}\,A$ there is an y so that $v = Ay$.

Why is $u+v$ also in $\mathrm{Col}\,A$?

$$u+v = Ax + Ay = A(x+y)$$

• linearity of the function represented by A

• since $x+y \in \mathbb{R}^n$ and $u+v = A(x+y)$ then $u+v \in \mathrm{Col}(A)$ by def of what the column space is.

Next the Nullspace ...

The null space of a matrix A is the set of all solutions to $Ax = 0$. The solutions of the homogeneous equation...

Suppose $Ax = 0$ and $Ay = 0$

then

$$\begin{array}{r} Ax = 0 \\ Ay = 0 \\ \hline Ax + Ay = 0 \end{array}$$

or $A(x+y) = 0$ so $x+y$ is a solution to the homogeneous equation..

Thus if $x, y \in \mathrm{Nul}(A)$ then $x+y \in \mathrm{Nul}(A)$

$$\boxed{\mathrm{Col}\,A = \{ Ax : x \in \mathbb{R}^n \}.}$$

a similar way to write $\mathrm{Nul}\,A$

$$\text{Nul } A = \{x : Ax = 0\}$$

One more definition before examples

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Want to write $H = \{Bx : x \in \mathbb{R}^p\}$

this matrix should have independent columns...

where $B = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ and the vectors v_i are linearly independent that is $Bx = 0$ has only the trivial solution or B has no free variables...

In particular, I want to see every subspace as the range of a matrix (linear function) where the matrix has independent columns.

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subspace of $\mathbb{R}^n$ and Col A is a subspace

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

Work these two examples from Section 2.8.

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$

What is the nullspace of $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$?

Solve $Ax = 0$

$$\begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix} \quad \begin{matrix} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 - 3r_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 2 & 1 & -5 \\ 0 & 2 & 4 & -8 \\ 0 & -4 & -8 & 16 \end{bmatrix} \quad r_3 \leftarrow r_3 + 2r_2$$

echelon form

$$U = \begin{bmatrix} 3 & 2 & 1 & -5 \\ 0 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Remember LU

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Solving $Ax=0$ $\quad LUx=0 \quad$ same as $Ux=0$

Make reduced echelon form

$$\begin{bmatrix} 3 & 2 & 1 & -5 \\ 0 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - r_2$$

$$\begin{bmatrix} 3 & 0 & -3 & 3 \\ 0 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} r_1 \leftarrow \frac{1}{3} r_1 \\ r_2 \leftarrow \frac{1}{2} r_2 \end{matrix}$$

(rescale pivots to 1.)

reduced echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

solve $Ax=0$

$$\begin{array}{cccc} P & P & F & F \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_3 + x_4 = 0 \\ x_2 + 2x_3 - 4x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = x_3 - x_4 \\ x_2 = -2x_3 + 4x_4 \end{cases}$$

Solution is now

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} x_4$$

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} x_4 \; : \; x_3, x_4 \in \mathbb{R} \right\}$$

Write this using a matrix of independent columns...

$$N = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

range of the linear function corresponding to N.

$$\mathrm{Nul}(A) = \{ N w : w \in \mathbb{R}^2 \} = \mathrm{col}(N)$$

To know this is a basis we need to verify that the columns of N are linearly independent. What that mean?

The only solution to $Nw = 0$ is the trivial $w=0$ solution.

$$N = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix}$$

this block already implies that the columns of N are linearly independent...

can't add a multiple of 0 to 1 and cancel the 1 out so no multiple of either vector is equal the other...

Thus this is a basis for Nul(A)...

The dimension of a subspace is the number of vectors in it's basis. Thus

$\dim \mathrm{Nul}(A) = 2$ since N has 2 columns.