choose pivot columns for basis ...

$$A = \begin{array}{c} \begin{array}{cccc} P & P & & \\ C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix} \end{array}$$

basis for Col A

$$(-1)\begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} = C_3$$

reduced echelon form of A:

$$\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ P & P & F & F \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$C_3 = -1\,C_1 + 2\,C_2$

$C_4 = 1\,C_1 - 4\,C_2$

Already know

$$\text{Nul } A = \{x : Ax = 0\} = \{Nw : w \in \mathbb{R}^2\} \text{ where } N = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What about Col A ?

basis matrix

$$\text{Col } A = \{Ax : x \in \mathbb{R}^4\} = \left\{ \begin{bmatrix} 3 & 2 \\ -9 & -4 \\ 9 & 2 \end{bmatrix} w : w \in \mathbb{R}^2 \right\}$$

Want to find a basis for this subspace. Since there are two free variables then 2 of these columns are dependent on the others.

dim Col A = 2 since 2 columns here

[rank A = 2]

The Rank Theorem $A \in \mathbb{R}^{m \times n}$

If a matrix A has n columns, then rank A + dim Nul $A = n$.

# of pivots + # of free vbls = # of variables

same as the number of columns.

The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

dim Col A = rank A

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

A basis for H is a set of vectors in H that

1. Span H
2. are linearly independent..

Theorem says if I know how many vectors are needed for a basis, then it's enough to check only one of the conditions...

- This all is based on the idea that for a square matrix having a pivot in every column (independence) implies there is a pivot in every row (spanning) and the other way around.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

Find a basis for Col A and a basis for Nul A.

find echelon and then reduced echelon form...

$$\begin{array}{ccc} p & p & \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}} \end{array}$$

Basis for column space are the pivot colums of A.

$r_2 \leftarrow r_2 - 4r_1$

$r_3 \leftarrow r_3 + 5r_1$

$r_4 \leftarrow r_4 - 2r_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 9 & 15 \\ 0 & 3 & 5 \end{bmatrix}$$

$r_3 \leftarrow r_3 + 3r_2$

$r_4 \leftarrow r_4 + r_2$

$$\begin{array}{ccc} p & p & f \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

already know dim Col A = # of pivots $= 2$
dim Nul A = # of free vbls $= \boxed{1}$

$$\text{Col } A = \left\{ \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ -5 & -1 \\ 2 & 7 \end{bmatrix}}_{} w : w \in \mathbb{R}^2 \right\}$$

basis matrix for column space.

To find Nul A we still need to solve $Ax = 0$

$$\text{Basis of Col } A = \left\{ \begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ 7 \end{bmatrix} \right\}$$

continue to find reduced echelon form

$$\begin{array}{ccc} p & p & \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_1 \leftarrow r_1 + \frac{2}{3} r_2$$

$$3 - \frac{2}{3} 5 = \frac{9-10}{3} = \frac{-1}{3}$$

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_2 \leftarrow -\frac{1}{3} r_2$$

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

algebraic form

$$x_1 - \tfrac{1}{3} x_3 = 0 \qquad x_1 = \frac{1}{3} x_3$$
$$x_2 + \tfrac{5}{3} x_3 = 0 \qquad x_2 = -\frac{5}{3} x_3$$

one free vbl.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} x_3 \\ -\frac{5}{3} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -5/3 \\ 1 \end{bmatrix} x_3$$

basis

$$\text{Nul } A = \left\{ \begin{bmatrix} 1/3 \\ -5/3 \\ 1 \end{bmatrix} w : w \in \mathbb{R} \right\}.$$

REM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\operatorname{rank} A = n$ dim Col A = n n pivots.... so invertible

p. $\dim \operatorname{Nul} A = 0$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

Section 2.9

11. $A = \begin{array}{c} \begin{array}{ccccc} P & P & F & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \end{array}$

echelon form of A $\sim \begin{array}{c} \begin{array}{ccccc} P & P & F & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$\operatorname{Col} A = \{Ax : x \in \mathbb{R}^5\} = \left\{ \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 4 \\ -3 & -9 & -7 \\ 3 & 10 & 11 \end{bmatrix} w : w \in \mathbb{R}^3 \right\}$

$\dim \operatorname{Col} A = 3$

Basis of Col A $= \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix} \right\}$

Find reduced echelon form

$\begin{array}{c} \begin{array}{ccccc} P & P & F & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$ $r_1 \leftarrow r_1 - 2r_2$

$$\begin{bmatrix} 1 & 0 & -9 & -8 & -11 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 + 8\, r_3$

$r_2 \leftarrow r_2 - 4\, r_3$

$$\begin{bmatrix} 1 & 0 & -9 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 9x_3 + 5x_5 = 0$

$x_2 + 2x_3 - 3x_5 = 0$

$x_4 + 2x_5 = 0$

$$x = \begin{bmatrix} 9x_3 - 5x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 \quad \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{basis of Nul } A} x_5$$

$$\text{Nul } A = \left\{ \begin{bmatrix} 9 & -5 \\ -2 & 3 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} w : \ w \in \mathbb{R}^2 \right\}$$

$\dim \text{Nul } A = 2$

Note $2 + 3 = 5$

$\dim \text{Col } A + \dim \text{Nul } A = \#$ of columns.