For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

assume $a_{11} \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

usual Gaussian elimination

$$r_2 \leftarrow r_2 - \frac{a_{21}}{a_{11}} r_1$$

$$r_3 \leftarrow r_3 - \frac{a_{31}}{a_{11}} r$$

Alternative...

$$r_2 \leftarrow a_{11} r_2$$

$$r_3 \leftarrow a_{11} r_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} a_{21} & a_{11} a_{22} & a_{11} a_{23} \\ a_{11} a_{31} & a_{11} a_{32} & a_{11} a_{33} \end{bmatrix}$$

$$r_2 \leftarrow r_2 - a_{21} r_1$$

$$r_3 \leftarrow r_3 - a_{31} r_1$$

these are now zero

assume $a_{11} a_{22} - a_{21} a_{12} \neq 0$ and do it again...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} a_{22} - a_{21} a_{12} & a_{11} a_{23} - a_{21} a_{13} \\ 0 & a_{11} a_{32} - a_{31} a_{12} & a_{11} a_{33} - a_{31} a_{13} \end{bmatrix}$$

$$r_3 \leftarrow (a_{11} a_{22} - a_{21} a_{12}) r_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{21}a_{12} & a_{11}a_{23}-a_{21}a_{13} \\ 0 & (a_{11}a_{22}-a_{21}a_{12})(a_{11}a_{32}-a_{31}a_{12}) & (a_{11}a_{22}-a_{21}a_{12})(a_{11}a_{33}-a_{31}a_{13}) \end{bmatrix}$$

$$r_3 \leftarrow r_3 - (a_{11}a_{32}-a_{31}a_{12})\, r_2$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{21}a_{12} & a_{11}a_{23}-a_{21}a_{13} \\ 0 & 0 & \underbrace{(a_{11}a_{22}-a_{21}a_{12})(a_{11}a_{33}-a_{31}a_{13}) - (a_{11}a_{32}-a_{31}a_{12})(a_{11}a_{23}-a_{21}a_{13})}_{\text{factor out } a_{11} \text{ from this... How?}} \end{bmatrix}$$

Terms missing a_{11}

$$a_{21}a_{12}a_{31}a_{13} - a_{31}a_{12}a_{21}a_{13} = 0$$

left with only the terms with a_{11} in them ...

$$a_{11}\Big(\underbrace{a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} - a_{21}a_{12}a_{33} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13} + a_{31}a_{12}a_{23}}_{\det A}\Big)$$

Solve by back substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{21}a_{12} & a_{11}a_{23}-a_{21}a_{13} \\ 0 & 0 & a_{11}\det A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\boxed{x_3 = \frac{c_3}{a_{11}\det A}}$$ ← Thus $\det A$ is in the denominator.

Section 3.3 develops Cramer's rule which allows writing a explicit formula for the solution of $Ax=b$ in term of determinants.

This theoretically useful to have an explicit formula for the solution, but it's such a complicated formula that its not useful for practical calculations (with numbers).

Notation given a matrix A the submatrix A_{ij} is obtained by crossing out the i^{th} row and j^{th} column of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \qquad A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

This formula makes sense now

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

could change those 1's to 2's or 3's and because of symmetries in the formula, same answer...

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= 1 \det \begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 1 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

please read Chapter 3 for next time...