a_{12}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

could change those 1's to 2's or 3's and because of symmetries in the formula, same answer...

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= 1 \det \begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 1 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

Determinant of a 2x2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

(in chapter 2 Finding inverses)

From chapter 2.2

THEOREM 4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

det A

If $ad - bc = 0$, then A is not invertible.
$\det(A) = 0$

generalize this in chapter 3.3 Cramer's rule

In terms of the recursive formula:

$$\det A = \sum_{j=1}^{\cancel{3}\,2} (-1)^{1+j} a_{1j} \det A_{1j}$$

a_{11}

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= a_{11}\det A_{11} - a_{12}\det A_{12}$$

$$= a\det[d] - b\det[c]$$

$$= ad - bc$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

could change those 1's to 2's or 3's and because of symmetries in the formula, same answer...

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= 1\det\begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 4 & 6 \\ 1 & 1 \end{bmatrix} + 3\det\begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

continue from last time

$$= 1(5\cdot 1 - 6\cdot 0) - 2(4\cdot 1 - 6\cdot 1) + 3(4\cdot 0 - 5\cdot 1)$$

$$= 5 + 4 - 15 = 9 - 15 = -6$$

Review find LU factor for A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

$r_2 \leftarrow r_2 - 4r_1$

$r_3 \leftarrow r_3 - r_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -2 & -2 \end{bmatrix}$$

$r_3' \leftarrow r_3 - \frac{2}{3} r_2$

$-2 - \frac{2}{3}(-6) = -2+4 = 2$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 2/3 & 1 \end{bmatrix}$$

easiest to do → multiply along the diagonal $1 \cdot (-3)(2) = -6$

not a coincidence

note if you swap rows, then first off the LU factorization gets mixed up... and every row swap need to be taken into account when finding the determinant as an additional (-1) factor...

Simpler $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ $\det A = 0 \cdot 3 - 1 \cdot 2 = -2$

By Gauss elimination

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad r_1 \leftrightarrow r_2$$

another (-1) because one row swap

echelon form

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$\det A = 1 \cdot 2 \cdot (-1)$

Question: why is multiplying along the diagonal in the echelon form and taking into account the row-swaps if needed the same as the recursive definition of det A ?

Example $\qquad A \in \mathbb{R}^{4\times 4}$

Gausian elimination to find the echelon form of A

$$\begin{bmatrix} 2 & 3 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \det A = 2\cdot 3\cdot 0\cdot 0\cdot \left(\begin{array}{c}\text{any } -1\text{'s} \\ \text{from row} \\ \text{swaps}\end{array}\right) = 0$$

Note its the diagonal of the matrix not the echelon.

Favorite formula for 3x3 matrix's

easiest to remember

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

1 works only for 3x3 matrices (and 2x2) but not 4x4, or higher... use this in Chapter 5 for solving the eigenvalue – eigenvector problem for 3x3 matrices.

2 add up all these numbers

$$\begin{array}{ccc} 12 & 0 & 5 \\ -0 & -8 & -15 \end{array}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} 12 \\ 5 \\ \hline 17 \\ -8 \\ \hline 9 \\ -15 \\ \hline \end{array}$$

$$\boxed{\det A = -6}$$

Why is the recursive formula so tedious when matrices get large?

Formula says...

It takes n determinants of size $(n-1)\times(n-1)$ to compute a determinant of size $n\times n$.

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

To find a 6×6 determinant

need 6 5×5 determinant.

needs 5 4×4 determinant

needs 4 3×3 determinant

3 2×2

2 1×1

Total work is $6\cdot 5\cdot 4\cdot 3\cdot 2 = 6!$ operations to finish... with recursive formula.

With Gaussian elimination about n^3 operations...