Compare the computational effort using the recursive definition with Gaussian elimination when computing the determinant:

n	recursive formula... $\sim n!$	Gaussian elimination... $\sim n^3$
1	1	1
2	2	8
3	6	27
4	24	64
5	120	125
6	720	216
7	5040	343

we'll use the recursive formula up to 3×3...

use Gaussian elimination here

& increases slower with size of matrix.

Properties of determinants:

Theorem 1 in section 3.1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

← could change the 1's for any other row...

In general fix the row i and expand along i

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

get the same answer no matter what row is chosen...

From before $i=1$ and we got:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

could change those 1's to 2's or 3's and because of symmetries in the formula, same answer...

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= 1 \det \begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 1 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

continue from last time

$$= 1(5\cdot 1 - 6\cdot 0) - 2(4\cdot 1 - 6\cdot 1) + 3(4\cdot 0 - 5\cdot 1)$$

$$= 5 + 4 - 15 = 9 - 15 = -6$$

Now try $i=3$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

a_{31}

whatever row you expand along you get the same answer.

$$\det A = \sum_{j=1}^{3} (-1)^{3+j} a_{3j} \det A_{3j}$$

$$= a_{31} \det A_{31} - a_{32} \det A_{32} + a_{33} \det A_{33}$$

$$= 1 \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} - \cancel{0 \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}} + 1 \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= 1 \cdot (2 \cdot 6 - 5 \cdot 3) + 1 \cdot (1 \cdot 5 - 4 \cdot 2) = 12 - 15 + 5 - 8 = -6$$

The theorem says you can expand along a column as well)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

where j is the fixed column being expanded.

Example

a_{12}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

Try $j = 2$

$$\det A = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} \det A_{i2}$$

$$= (-1)^3 a_{12} \det A_{12} + (-1)^4 a_{22} \det A_{22} + (-1)^5 a_{32} \det A_{32}$$

$$= -2 \det \begin{bmatrix} 4 & 6 \\ 1 & 1 \end{bmatrix} + 5 \det \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

$$= -2\left(4\cdot 1 - 6\cdot 1\right) + 5\left(1\cdot 1 - 3\cdot 1\right) = -2(-2) + 5(-2) = -6$$

Why is Theorem 1 true? We're not going to spend a lot of time on this...

What are the consequences?

expansion along row i

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{for any } i \text{ fixed}$$

expansion along column j

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{for any } j \text{ fixed}$$

Remember A^T switches the columns with the rows...

Thus $\det A = \det A^T$

Suppose A is upper triangular... then

$$\det A = \underbrace{a_{11} a_{22} \cdots a_{nn}}$$

product along diagonal of the triangular matrix...

Why... because just expand along the first column:

the only a_{i1} that's not zero.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$\det A = 1 \cdot 5 \cdot 8 \cdot 10 = 400$

why?

expand on this column

$$\det A = \sum_{i=1}^{4} (-1)^{i+1} a_{i1} \det A_{i1} = (-1)^{1+1} a_{11} \det A_{11}$$

$$= 1 \cdot \det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}$$

expand on this column

$$= 1 \cdot \left(5 \det \begin{bmatrix} 8 & 9 \\ 0 & 10 \end{bmatrix} \right) = 1 \cdot 5 \cdot 8 \cdot \det 10$$

$$= 1 \cdot 5 \cdot 8 \cdot 10 = 400$$

The same hold true for an lower triangular matrix.

Why?, Either expand along the first row, or take the transpose and compute $\det A^T$ where A^T is now upper triangular.

To use Gaussian elimination to compute the determinant we need to know how the row operations affect the determinant...

Elementary Row Operations.

① $r_i \leftarrow r_i - \alpha r_j \quad i \neq j$

② $r_i \leftrightarrow r_j \quad i \neq j$

③ $r_i \leftarrow \alpha r_i \quad \alpha \neq 0$

How is the det A related to whatever you get after performing one of these operation?

properties of determinants

matrix corresponding to the row operation

① $\det([r_i \leftarrow r_i - \alpha r_j]\, A) = \det A$

② $\det([r_i \leftrightarrow r_j]\, A) = -\det A$

③ $\det([r_i \leftarrow \alpha r_i]\, A) = \alpha \det A$