…ies
…minants

↙ matrix corresponding to the row operation

① $\det([r_i \leftarrow r_i - \alpha r_j] A) = \det A$

② $\det([r_i \leftrightarrow r_j] A) = -\det A$

③ $\det([r_i \leftarrow \alpha r_i] A) = \alpha \det A$

matrix...

Example

$r_2 \leftarrow \frac{1}{2} r_2$ what the matrix the corresponds to this row operation?

Perform the operation on the identity matrix to figure out what matrix for the operation is...

Note to find the matrix for any linear functions f we just needed to know $f(e_1), f(e_2), \ldots, f(e_n)$ to write the matrix corresponding to f.

note $e_1, e_2, \ldots, e_n$ are just the columns of I

$$f(x) = Ax \qquad A = \left[f(e_1) \middle| f(e_2) \middle| \cdots \middle| f(e_n) \right]$$

Thus we apply $r_2 \leftarrow \frac{1}{2} r_2$ to the identity matrix

$n=3$, 3×3 case

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \text{ then } \left[r_2 \leftarrow \tfrac{1}{2}r_2\right] = \begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix}$$

...es
...minants

matrix corresponding to the row operation

① $\det\left(\underbrace{[r_i \leftarrow r_i - \alpha r_j]}_{\text{elementary matrix}} A\right) = \det A$

② $\det\left([r_i \leftrightarrow r_j] A\right) = -\det A$

③ $\det\left(\overbrace{[r_i \leftarrow \alpha r_i]}^{\text{matrix}} A\right) = \alpha \det A$ matrix

This is talking about determinants and how they interact with matrix multiplication.

$$\det\left[r_2 \leftarrow \tfrac{1}{2}r_2\right] = \det\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix} = 1\cdot\tfrac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\det\left(\overbrace{[r_i \leftarrow \underset{\frac{1}{2}}{\alpha} r_i]}^{\text{matrix}} A\right) = \underset{\frac{1}{2}}{\alpha} \det A$$

rewrite as

$$\det\left([r_i \leftarrow \alpha r_i] A\right) = \det\left([r_i \leftarrow \alpha r_i]\right) \det A$$

The other elementary row operations have the same form matrix corresponding to the row operati[on]

① $\det([r_i \leftarrow r_i - \alpha r_j]\, A) = 1 \cdot \det A$

Example $n=3$, $r_2 \leftarrow r_2 + 3r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [r_2 \leftarrow r_2 + 3r_1] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det [r_2 \leftarrow r_2 + 3r_1] = \det \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \cdot 1 \cdot 1 = 1$$

Thus

$$\det([r_i \leftarrow r_i - \alpha r_j] A) = \det([r_i \leftarrow r_i - \alpha r_j]) \det A$$

Same thing happens for row swaps. Thus, if E is a matrix corresponding to any row operation, then

$$\det(EA) = (\det E)(\det A)$$

Elementary matrices are used to create the LU factorization as well as the reduced echelon form..

Theorem: $\det AB = \det A \det B$
for any square matrices $A, B \in \mathbb{R}^{n \times n}$.

Suppose A not invertible ... then the reduced row echelon form has a zero row... i.e. it's missing a pivot.

det A = ± product along the diagonal of the echelon form. = 0

If A is invertible, then the reduced echelon form of A is I.

maybe $\frac{1}{2}r_1$, $r_1 \leftarrow \frac{1}{2}r_1$

row operations for example $r_2 \leftarrow r_2 - \frac{1}{2}r_1$

$$I = \underbrace{E_p \cdots E_3 E_2 E_1}_{\text{all these row operations are invertible}} A$$

$$E_p^{-1} I = \cancel{E_p^{-1}} \cancel{E_p} \cdots E_3 E_2 E_1 A$$

$$E_{p-1}^{-1} E_p^{-1} = \cancel{E_{p-1}^{-1}} \cancel{E_{p-1}} \cdots E_3 E_2 E_1 A$$

$$\vdots$$

$$E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} = A$$

Conclusion an (square) invertible matrix can be written as a product of elementary row operations. I.e, the inverses of the row operations needed to create the reduced row echelon form in reverse order...

We know that $\det(EB) = (\det E)(\det B)$ for any elementary matrix E and any square B.

We know any invertible matrix A can be written as a product of elementary matrices...

Thus $\det AB = (\det A)(\det B)$.

Why?

$$
\begin{aligned}
\det AB &= \det\Big(\overbrace{\underbrace{E_1^{-1}}_{\text{elementary}}\, E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}}^{A}\ B\Big) \qquad \text{(}E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B \text{ : some matrix)}\\
&= \det(E_1^{-1}) \det\left(E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B\right) \\
&= \det(E_1^{-1}) \det(E_2^{-1}) \det\left(E_3^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B\right) \\
&= \det(E_1^{-1}) \det(E_2^{-1}) \cdots \underbrace{\det(E_{p-1}^{-1}) \det(E_p^{-1})}_{\text{combine}} \det B \\
&= \det(E_1^{-1}) \det(E_2^{-1}) \cdots \det(E_{p-1}^{-1} E_p^{-1}) \det B \\
&= \det\left(E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}\right) \det B = \det A \det B
\end{aligned}
$$

all the elementary matrices are out

put the elementary matrices back together

Thus... $\det AB = \det A \det B$

when A is invertible...

What about when A is not invertible... already we know that means $\det A = 0$.

what $\det AB$? If A (square matrix) is not invertible then AB is also not invertible (so $\det AB = 0$)

also $(\det A)(\det B) = 0 \cdot \det B = 0$

- If A is not invertible (but it is square) then it must be missing pivots... this means AB is also missing pivots...

Example: of a ~~non-square~~ matrix $A \in \mathbb{R}^{m \times n}$ with $m \neq n$ and $B \in \mathbb{R}^{n \times m}$, such that AB is invertible (For next time).

Thus A and B were not invertible, but their product is invertible...

So $\det AB = (\det A)(\det B)$ always, whether A is invertible or not