Finishing Chapter 3. Next chapter is Chapter $\boxed{6.}$

we'll come back to chapter 5 and then finish with 7.

Cramer's Rule:

Definition of determinant $A \in \mathbb{R}^{n \times n}$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ expanding on column j

or

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ expanding on row i

$$A = \begin{array}{c} \begin{array}{cccc} & & j=3 & \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \end{array}$$

$(-1)^5 = -1$

$$\det A = a_{13} \det A_{13} - a_{23} \det A_{23} + a_{33} \det A_{33} - a_{43} \det A_{43}$$

$$= \underbrace{\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}}_{\text{column of } A} \cdot \underbrace{\begin{bmatrix} \det A_{13} \\ -\det A_{23} \\ \det A_{33} \\ -\det A_{43} \end{bmatrix}}_{\text{something else}}$$

The determinant of A is equal to the dot product of a column of A with something else

Conclusion is that $\det A$ is a linear function of any column.

That is,

$$\det A = f\left(\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}\right) = \begin{bmatrix} \det A_{13} & -\det A_{23} & \det A_{33} & -\det A_{43} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & \overset{a_{12}}{3} & 0 \\ 2 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

I know det A is a linear function of that column...

$$\det A = -3\underbrace{\det\begin{bmatrix} 2 & 0 \\ 5 & 7 \end{bmatrix}}_{14} + 4\underbrace{\det\begin{bmatrix} 1 & 0 \\ 5 & 7 \end{bmatrix}}_{7} - 6\underbrace{\det\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}_{0}$$

$$= \begin{bmatrix} -14 & 7 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = -42 + 28 = -14.$$

$$\begin{array}{r} 42 \\ 28 \\ \hline 14 \end{array}$$

PROOF Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and the columns of the $n \times n$ identity matrix I by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. If $A\mathbf{x} = \mathbf{b}$, the definition of matrix multiplication shows that

$$A\,(I_i(\mathbf{x})) = A\begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} = A_i(\mathbf{b})$$

what is that?

By the multiplicative property of determinants,

$$(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$$

The second determinant on the left is simply x_i. (Make a cofactor expansion along the ith row.) Hence $(\det A)\,x_i = \det A_i(\mathbf{b})$. This proves (1) because A is invertible and $\det A \neq 0$. ■

$I_i(x)$ ← Means take the matrix I and replace the ith column with the vector x.

Suppose $n=3$, then

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_2(x) = \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}$$

Q: What is $\det I_2(x)$?

Seems reasonable to expand on the second column... BUT even better expand the 2^{nd} row...

$$\det\begin{bmatrix}1 & x_1 & 0\\ 0 & x_2 & 0\\ 0 & x_3 & 1\end{bmatrix} = -0\det\begin{bmatrix}?\end{bmatrix} + x_2\det\begin{bmatrix}1&0\\0&1\end{bmatrix} - 0\det\begin{bmatrix}?\end{bmatrix} = x_2$$

(2,1)

Q: What is $\det I_3(x)$? expand on 3^{rd} row

$$\det\begin{bmatrix}1 & 0 & x_1\\ 0 & 1 & x_2\\ 0 & 0 & x_3\end{bmatrix} = 0\det\begin{bmatrix}?\end{bmatrix} - 0\det\begin{bmatrix}?\end{bmatrix} + x_3\det\begin{bmatrix}1&0\\0&1\end{bmatrix} = x_3$$

(3,1)

Q: What is $A I_2(x)$?

standard basis vectors...

$$A I_2(x) = A\begin{bmatrix}1 & x_1 & 0\\ 0 & x_2 & 0\\ 0 & x_3 & 1\end{bmatrix} = A\begin{bmatrix} e_1 & x & e_3\end{bmatrix} = \begin{bmatrix} Ae_1 & Ax & Ae_3\end{bmatrix}$$

If trying to solve $Ax=b$

$$= \begin{bmatrix} Ae_1 & b & Ae_3\end{bmatrix} = \begin{bmatrix} a_1 & b & a_3\end{bmatrix} = A_2(b)$$

$$A I_2(x) = A_2(b)$$

If $A = \begin{bmatrix} a_1 & a_2 & a_3\end{bmatrix}$ then

$$\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3\end{bmatrix}\begin{bmatrix} a_1\end{bmatrix}$$

Cramer's rule puts this all together...

$A I_j(x) = A_j(b)$ for any column j

$\det(A I_j(x)) = \det A_j(b)$

$(\det A)(\det I_j(x)) = \det A_j(b)$

$(\det A)\, x_j = \det A_j(b)$

$$\boxed{x_j = \frac{\det A_j(b)}{\det A}}$$

Cramer's rule...

Note: The advantage of Cramer's rule is that its an explicit formula for the solution of $Ax=b$.

This is not an efficient way to find x, but it could have theoretical uses...

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Use cramer's rule to find an explicit formula for A^{-1}.

To know the matrix corresponding to A^{-1} I need to know

$$\boxed{Av_1 = e_1 \quad Av_2 = e_2, \quad \ldots, \quad Av_n = e_n}$$

solve all of these using Cramer's rule to find A^{-1} explicitly...

Then $A^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.

Example $n=3$

$$A^{-1} = \begin{bmatrix} \det A_1(e_1) & \det A_1(e_2) & \det A_1(e_3) \\ \det A_2(e_1) & \det A_2(e_2) & \det A_2(e_3) \\ \det A_3(e_1) & \det A_3(e_2) & \det A_3(e_3) \end{bmatrix} \Big/ \det A$$

almost done, but these determinants can be simplified...

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A_2(e_3) = \begin{array}{c} \begin{array}{ccc} & e_3 & \end{array} \\ \begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33} \end{vmatrix} \end{array} = -1 \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

column

(32)

expand on 2^{nd} column.

these turn out to be the cofactors of A