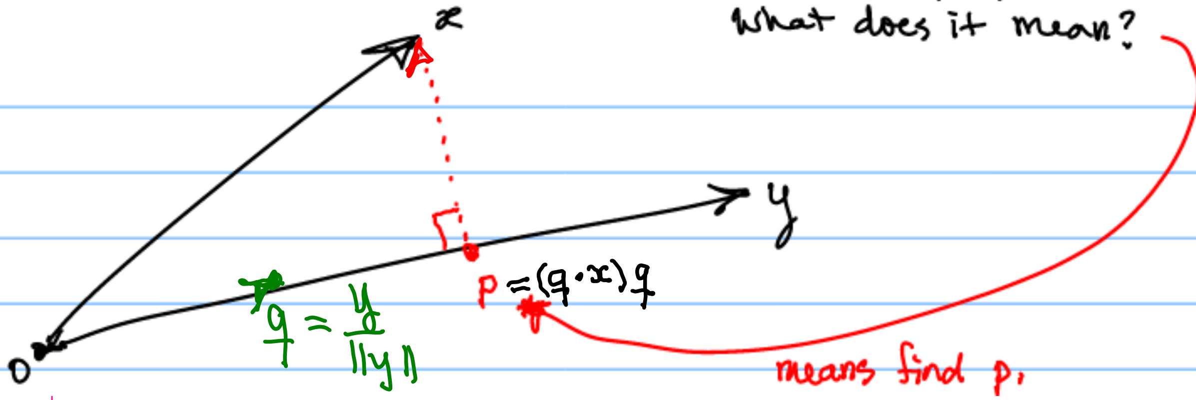
want to project $x$ onto $y$.
What does it mean?




Let $p = ty$

Idea solve for $t$ so the vector $x - p$ is perpendicular to $y$.

means dot product

$$(x-p)\cdot y = 0$$

$$(x - ty)\cdot y = 0$$

$$x\cdot y - t\, y\cdot y = 0$$

$$t = \frac{x\cdot y}{y\cdot y} = \frac{y\cdot x}{y\cdot y}$$

Thus

$$p = \frac{y\cdot x}{y\cdot y}\, y$$

recall $y\cdot y = \|y\|^2$

$$p = \frac{y\cdot x}{\|y\|^2}\, y = \left(\frac{y}{\|y\|}\cdot x\right)\frac{y}{\|y\|}$$

this is the unit vector pointing in the $y$ direction.

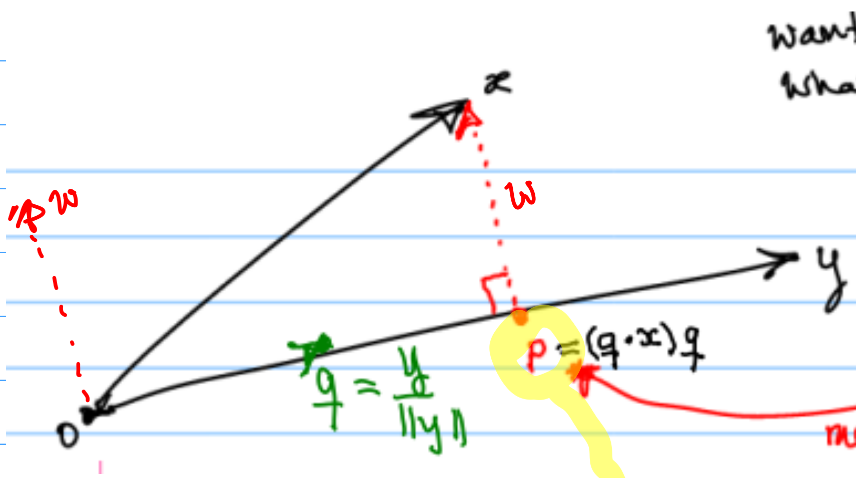
unit vector $q = \frac{y}{\|y\|}$

Note sometime the unit vector in the $y$ direction is written $\hat{y}$, but not in our book...

The projection $p$ in terms of the unit vector $q$ is simply

$$p = (q\cdot x)\,q$$

Thus the vector $x$ is written as two orthogonal components:




$$q\cdot q = \frac{y}{\|y\|}\cdot\frac{y}{\|y\|} = 1$$

$$x = p + \underbrace{(x-p)}_{w} = p + w \quad \text{where} \quad p\cdot w = 0$$

$$p\cdot w = p\cdot(x-p) = p\cdot x - p\cdot p$$

$$= p\cdot x - p\cdot\big((q\cdot x)q\big) =$$

$$= (q\cdot x)q\cdot x - (q\cdot x)\overbrace{q\cdot\big((q\cdot x)q\big)}$$

$$= (q\cdot x)^2 - (q\cdot x)^2\, q\cdot q = 0.$$

Theorem 3 in Chapter 6.1

Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the null space of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad \boxed{(\text{Col } A)^{\perp} = \text{Nul } A^T}$$

what's that?

That's the orthogonal complement.

Let $W \subseteq \mathbb{R}^n$ then

$$W^{\perp} = \{ z \in \mathbb{R}^n : z \cdot w = 0 \text{ for all } w \in W \}$$

View theorem in two way:

① a way to compute $W^{\perp}$

② an interesting relation between Column spaces and Nulspaces.

Why is $\boxed{(\text{Col } A)^{\perp} = \text{Nul } A^T}$ ?

$A \in \mathbb{R}^{m \times n}$

$A^T \in \mathbb{R}^{n \times m}$

What is Col A? Nul $A^T$ ?

$w \in$ $\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

$\text{Nul } A^T = \{ y \in \mathbb{R}^m : A^T y = 0 \}$

Now

$$(\text{Col } A)^{\perp} = \{ z \in \mathbb{R}^m : z \cdot w = 0 \text{ for all } w \in \text{Col } A \}$$

Note $w \in \text{Col } A$ means $w = Ax$ for some $x \in \mathbb{R}^n$

$$= \{ z \in \mathbb{R}^m : z \cdot Ax = 0 \text{ for all } x \in \mathbb{R}^n \}$$

$$= \{ z \in \mathbb{R}^m : A^T z \cdot x = 0 \text{ for all } x \in \mathbb{R}^n \}$$

$A^T z \cdot x = 0$ for all $x \in \mathbb{R}$ means that $A^T z = 0$ why?

Taking $x = A^T z$ implies
$A^T z \cdot A^T z = 0$
$\|A^T z\|^2 = 0$ so $A^T z = 0$

$$(\text{Col } A)^{\perp} = \{ z \in \mathbb{R}^m : A^T z = 0 \} = \text{Nul } A^T$$

So $(\text{Col } A)^{\perp} = \text{Nul } A^T$

How could you use this?

Suppose I have a subspace $W$ and I want to compute $W^{\perp}$.

① Find a basis for $W$
$\{v_1, v_2, \dots, v_n\}$

(2) $A = \left[ v_1 \middle| v_2 \middle| \cdots \middle| v_n \right]$

put basis vectors as columns in $A$

Thus $\operatorname{Col} A = W$ by what we just did

(3) $W^\perp = (\operatorname{Col} A)^\perp = \operatorname{Nul} A^T$

compute this to find $W^\perp$.