The book writes the Gram-Schmidt without introducing the unit vectors q. This avoids square roots in the algorithm, but make the matrix Q and R more difficult to identify

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$v_2 = x_2 - \left(\underbrace{\frac{v_1}{\|v_1\|}}\cdot x_2\right)\underbrace{\frac{v_1}{\|v_1\|}}$$

we identified $q_1 = \dfrac{v_1}{\|v_1\|}$ unit vector

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p \tag{1}$$

start with v's since x's are usually the variable rather than columns of A.

Let $\{\underbrace{V_1, V_2, \ldots, V_n}_{\text{columns of } A}\}$ be a basis for W. Then

$z_1 = V_1$ $\qquad q_1 = \dfrac{z_1}{\|z_1\|}$

$z_2 = V_2 - (q_1 \cdot V_2)q_1$ $\qquad q_2 = \dfrac{z_2}{\|z_2\|}$

$z_3 = V_3 - (q_1 \cdot V_3)q_1 - (q_2 \cdot V_3)q_2$ $\qquad q_3 = \dfrac{z_3}{\|z_3\|}$

$\vdots$ $\qquad \vdots$

columns of Q

$$z_n = V_n - (q_1 \cdot V_n)q_1 - (q_2 \cdot V_n)q_2 - \cdots - (q_{n-1} \cdot V_n)q_{n-1}$$

work these Examples!

$$q_n = \frac{z_n}{\|z_n\|}$$

Chapter 6.4

3. $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$ next time...

first

3. $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}}_{V_1} \quad \underbrace{\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}}_{V_2}$

$$\|z_1\| = \sqrt{2^2 + (-5)^2 + 1^2} = \sqrt{4+25+1} = \sqrt{30}$$

$$z_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \qquad q_1 = \frac{z_1}{\|z_1\|} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

need to know this coefficient to find R

$$z_2 = V_2 - (q_1 \cdot V_2) q_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \underbrace{\left(\frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right)}_{\text{need to know this coefficient to find R}} \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \left(\frac{1}{\sqrt{30}} (8+5+2) \right) \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{\sqrt{30}} \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$q_2 = \frac{z_2}{\|z_2\|} = \frac{1}{\frac{1}{2}\sqrt{54}} \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} = \frac{1}{\sqrt{54}} \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

This is one of the coefficients in R.

$$-1 + \frac{5}{2} = \frac{5}{2} - \frac{2}{2} = \frac{3}{2}$$

Turning this into a unit vector is the same as turning any multiple into a unit vector, so one could use $\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$ to avoid the fractions. However

$$\begin{array}{r} 36 \\ 9 \\ \hline 45 \\ 9 \\ \hline 54 \end{array}$$

$$\left\| \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \right\| = \frac{1}{2} \sqrt{36+9+9} = \frac{1}{2}\sqrt{54}$$

Thus starting with the vectors $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

I used Gram-Schmidt to obtain

$$\frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{54}}\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ -5 & -1 \\ 1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 2/\sqrt{30} & 6/\sqrt{54} \\ -5/\sqrt{30} & 3/\sqrt{54} \\ 1/\sqrt{30} & 3/\sqrt{54} \end{bmatrix}$$

These columns are a linear combination of the columns of A.

Since the column operations were reversible, then the columns of A are also linear combinations of the columns of Q.

Therefore $\text{Col } A = \text{col } Q$

Claim: $A = QR$ for some upper triangular matrix R.

R: correspond to the reverse of the column operations used to obtain Q from A.

Recall when we did row operations for Gaussian elimination we ended up with $A = LU$

L: corresponded to the reverse of the row operations used to obtain U from A.

Note column operations are mult on the right by R

row operations are mult on the left by L

$$z_1 = v_1 = \|z_1\| q_1 \qquad q_1 = \frac{z_1}{\|z_1\|}$$

$$z_2 = v_2 - (q_1 \cdot v_2) q_1 \qquad q_2 = \frac{z_2}{\|z_2\|}$$

$$v_2 = z_2 + (q_1 \cdot v_2) q_1$$

$$v_2 = \|z_2\| q_2 + (q_1 \cdot v_2)$$

Thus :

$$v_1 = \|z_1\| q_1$$

$$\underbrace{v_2 = (q_1 \cdot v_2) q_1 + \|z_2\| q_2}$$

write this as matrix multiplication

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \|z_1\| q_1 & (q_1 \cdot v_2) q_1 + \|z_2\| q_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \underbrace{\begin{bmatrix} \|z_1\| & q_1 \cdot v_2 \\ 0 & \|z_2\| \end{bmatrix}}_{R}$$

$$R = \begin{bmatrix} \|z_1\| & q_1 \cdot v_2 \\ 0 & \|z_2\| \end{bmatrix} = \begin{bmatrix} \sqrt{30} & 15/\sqrt{30} \\ 0 & \frac{1}{2}\sqrt{54} \end{bmatrix} \qquad \frac{15}{\sqrt{30}} = \frac{15\sqrt{30}}{30} = \frac{1}{2}\sqrt{30}$$

Therefore $A = QR$ where ...

$$\begin{bmatrix} 2 & 4 \\ -5 & -1 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/\sqrt{30} & 6/\sqrt{54} \\ -5/\sqrt{30} & 3/\sqrt{54} \\ 1/\sqrt{30} & 3/\sqrt{54} \end{bmatrix}} \begin{bmatrix} \sqrt{30} & \frac{1}{2}\sqrt{30} \\ 0 & \frac{1}{2}\sqrt{54} \end{bmatrix}$$

Note the columns of Q are orthonormal vectors...

$$q_1 \cdot q_1 = 1, \quad q_1 \cdot q_2 = 0, \quad q_2 \cdot q_2 = 1$$

Thus $Q^TQ = I$ ← explain this more next time... note Q is not invertible since not square, but can undo its effects by mult. on left with Q^T.