rather than columns of A.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for W. Then

columns of A

$z_1 = v_1$ $\qquad q_1 = \dfrac{z_1}{\|z_1\|}$

look simpler

$z_2 = v_2 - (q_1 \cdot v_2) q_1$ since q is a unit vector $\qquad q_2 = \dfrac{z_2}{\|z_2\|}$

$z_3 = v_3 - (q_1 \cdot v_3) q_1 - (q_2 \cdot v_3) q_2$ $\qquad q_3 = \dfrac{z_3}{\|z_3\|}$

$\vdots$

columns of Q

$$z_n = v_n - (q_1 \cdot v_n) q_1 - (q_2 \cdot v_n) q_2 - \cdots - (q_{n-1} \cdot v_n) q_{n-1} \qquad q_n = \frac{z_n}{\|z_n\|}$$

work these Examples!

$A = QR$ finding

Q on the left

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ \textbf{10.} & \left[\begin{matrix} -1 \\ 3 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 6 \\ -8 \\ -2 \\ -4 \end{matrix} & \left.\begin{matrix} 6 \\ 3 \\ 6 \\ -3 \end{matrix}\right] \end{array}$$

$$z_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad q_1 = \frac{z_1}{\|z_1\|} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\|z_1\| = \sqrt{1+9+1+1} = \sqrt{12} = 2\sqrt{3}$$

$$Q = \begin{bmatrix} \frac{-1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \end{bmatrix}, \qquad R = \begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

(1,2) slot

q_1 on the left in dot product

$$z_2 = v_2 - (q_1 \cdot v_2) q_1$$

row column of the R matrix

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \underbrace{\left(\frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \right)}_{\text{goes in the R matrix}} \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} = -6 - 24 - 2 - 4 = -36$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{1}{2\sqrt{3}} (-36) \right) \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$-6\sqrt{3} = \frac{-18}{\sqrt{3}}$$

$$z_2 = \begin{bmatrix}6\\-8\\-2\\-4\end{bmatrix} - \left(\frac{-18}{\sqrt{3}}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}6\\-8\\-2\\-4\end{bmatrix} + 3\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} \qquad q_2 = \frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$z_3 = V_3 - (q_1 \cdot V_3)q_1 - (q_2 \cdot V_3)q_2$$

$$= \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(\frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} = -6+9+6-3 = 6 \quad \text{and} \quad \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} = 18+3+6+3 = 30$$

Therefore,

$$z_3 = \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\frac{6}{2\sqrt{3}}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(\frac{30}{2\sqrt{3}}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\sqrt{3}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(5\sqrt{3}\right)\frac{1}{2\sqrt{3}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \frac{5}{2}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix} \qquad q_3 = \frac{1}{2\sqrt{3}}\begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix}$$

Therefore $A = QR$ where

$$Q = \begin{bmatrix} \frac{-1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \end{bmatrix}, \quad R = \begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

Note:

$$Q = \frac{1}{2\sqrt{3}}\begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

This denominator is still part of each column.

Checked $Q^TQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Try to solve $Ax = b$ ... plug in $A = QR$.

$$QRx = b$$

$$Q^TQRx = Q^Tb$$

Thus,... $Rx = Q^Tb$ can be solved by substitution because R is triangular.

This solves $Ax = b$ is a way that minimizes the residual error... thus it's stable with respect to rounding errors and provides the least squares minimizer when $Ax = b$ doesn't have a solution...