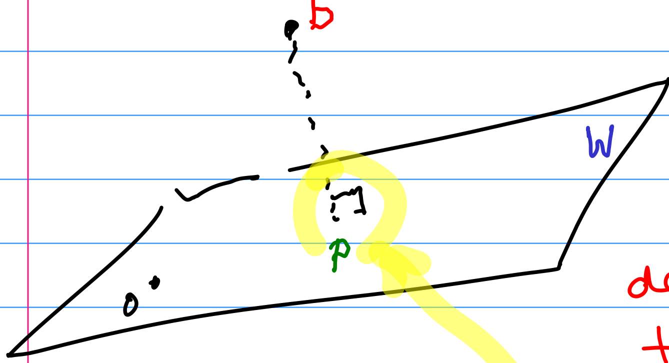
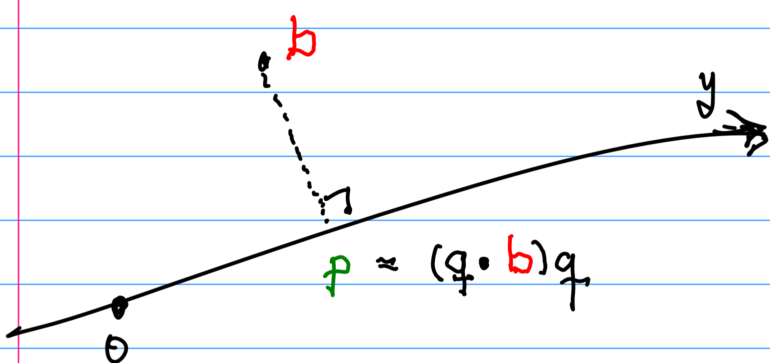



The closest point p in the plane to b

describe plane passing through the origin as $W = \text{Col}\, A$

The right angle means $b - x$ is perpendicular to any vector in the plane... $b - x \in W^\perp$

Already done this




Closest point on the line.

$$q = \frac{y}{\|y\|}$$

$$W^\perp = \{ z : z \cdot w = 0 \text{ for all } w \in W \}$$

Let $A$ be an $m \times n$ matrix. The orthogonal complement of the row space of $A$ is the null space of $A$, and the orthogonal complement of the column space of $A$ is the null space of $A^T$:

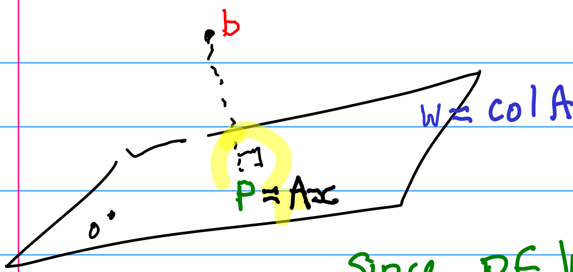
$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad \boxed{(\underbrace{\text{Col } A}_{})^\perp = \text{Nul } A^T}$$

$\text{Col}\, A = W$

$W^\perp = \text{Nul}\, A^T$

$$\text{Nul}\, A^T = \{ u : A^T u = 0 \}$$

Find $x$ so that $b - p \in (\mathrm{Col}\, A)^{\perp} = \mathrm{Nul}\, A^T$




Thus $b - p = u$ such that $A^T u = 0$

Thus $A^T(b - p) = 0$

Since $p \in W = \mathrm{col}\, A$

then $p = Ax$ for some $x$

Thus $A^T(b - Ax) = 0$

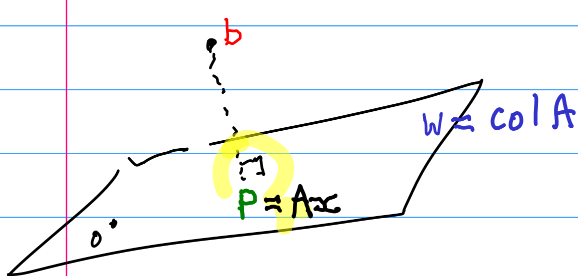
$A^T b - A^T A x = 0$

or $\boxed{A^T A x = A^T b}$

normal equations...

Solution to the normal equations minimizes the distance between b and Ax. In other words it minimized $\|Ax - b\|$.

residual error when plugging $x$ into the equation $Ax = b$.




This is reasonable to do even when $Ax = b$ doesn't have a solution (as in the picture)

Try it:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find the least squares "solution" to $Ax = b$, that is the value of $x$ for which $\|Ax - b\|$ is minimal.

Solve the normal equations $A^T A x = A^T b$

Not factoring $A^T A$

$$A^T b = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix}$$

Now solve

$$\begin{bmatrix} 14 & 10 \\ 10 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cc:c} 14 & 10 & 1 \\ 10 & 14 & 3 \end{array}\right]$$

$r_2 \leftarrow r_2 - \frac{5}{7} r_1$

$$\begin{array}{r} 14 \\ \times\ 7 \\ \hline 98 \\ -50 \\ \hline 48 \end{array}$$

$$\left[\begin{array}{cc:c} 14 & 10 & 1 \\ 0 & \frac{48}{7} & \frac{16}{7} \end{array}\right]$$

$14x_1 + 10x_2 = 1$

$\frac{48}{7}x_2 = \frac{16}{7}$ $\qquad x_2 = \frac{16}{48} = \frac{8}{24} = \frac{4}{12} = \frac{1}{3}$

$x_1 = \frac{1 - 10x_2}{14} = \frac{3-10}{3 \cdot 14} = \frac{-7}{3 \cdot 14} = \frac{-1}{6}$

$x = \begin{bmatrix} -1/6 \\ 1/3 \end{bmatrix}$ is the value of $x$ that minimizes $\|Ax - b\|$.

A better way is to use Gram-Schmidt

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}$ $\qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A^TAx = A^Tb$

Not factoring $A^TA$ but instead $A = QR$

$Q^TQ = I$

$(QR)^T QRx = (QR)^T b$

$R^T \cancel{Q^T} \cancel{Q} Rx = R^T Q^T b$

$$\cancel{R^T} Rx = \cancel{R^T} Q^T b$$

like to cancel the $R^T$ ... is that possible?

What is $R$? Came from Gram-Schmidt

$$R = \begin{bmatrix} \|z_1\| & q_1 \cdot v_2 & \cdots & q_1 \cdot v_n \\ 0 & \|z_2\| & & \vdots \\ & & \ddots & \\ 0 & 0 & \cdots & \|z_n\| \end{bmatrix}$$

note these values were in the denominators $q_i = \frac{z_i}{\|z_i\|}$

So the diagonal entries of $R$ are not zero, otherwise the Gram-Schmidt algorithm wouldn't have worked... So $R$ is invertible..

Note $R^T$ has the same non-zero diagonal entries... so it's invertible too...

Therefore $(R^T)^{-1}$ exists...

$$\cancel{(R^T)^{-1}} \cancel{R^T} Rx = \cancel{(R^T)^{-1}} \cancel{R^T} Q^T b$$

Conclusion: Solving $Rx = Q^T b$ where $A = QR$ is the QR factorization of $A$ minimizes $\|Ax - b\|$.