$A^TAx = A^Tb$

Not factoring A^TA but instead $A = QR$

$Q^TQ = I$

$$(QR)^T(QR)x = (QR)^T b$$

$$R^T \cancel{Q^TQ} Rx = R^T Q^T b$$

$$\cancel{R^T} Rx = \cancel{R^T} Q^T b$$

under the assumption that R^T is invertible...

In our case we took the columns of A to be a basis of W, so yes they are independent...

Solve $Rx = Q^Tb$ to minimize $\|Ax - b\|$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

v_1 v_2

Note the solution is

$$\boxed{x = \begin{bmatrix} -1/6 \\ 1/3 \end{bmatrix}}$$

Find the QR factorization of the matrix A...

Note sometimes the factors Q and R are given already in the HW.

Use Gram-Schmidt to find Q and R...

$$Q = \begin{bmatrix} 1/\sqrt{14} & 4/\sqrt{21} \\ 2/\sqrt{14} & 1/\sqrt{21} \\ 3/\sqrt{14} & -2/\sqrt{21} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{14} & 10/\sqrt{14} \\ 0 & \frac{4}{7}\sqrt{21} \end{bmatrix}$$

$$z_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad q_1 = \frac{z_1}{\|z_1\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$z_2 = v_2 - (q_1 \cdot v_2)q_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{10}{\sqrt{14}}\right)\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \frac{10}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16/7 \\ 4/7 \\ -8/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16 \\ 4 \\ -8 \end{bmatrix} = \frac{4}{7}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \qquad q_2 = \frac{z_2}{\|z_2\|} = \frac{1}{\frac{4}{7}\sqrt{21}}\,\frac{4}{7}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{21}}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

Now want to solve $\boxed{Ax = b}$ using $A = QR$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solve $\boxed{Rx = Q^T b}$

$$Q = \begin{bmatrix} 1/\sqrt{14} & 4/\sqrt{21} \\ 2/\sqrt{14} & 1/\sqrt{21} \\ 3/\sqrt{14} & -2/\sqrt{21} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{14} & 10/\sqrt{14} \\ 0 & \frac{4}{7}\sqrt{21} \end{bmatrix}$$

$$Q^T b = \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ 4/\sqrt{21} & 1/\sqrt{21} & -2/\sqrt{21} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 4/\sqrt{21} \end{bmatrix}$$

$$Rx = \begin{bmatrix} \sqrt{14} & 10/\sqrt{14} \\ 0 & \frac{4}{7}\sqrt{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 4/\sqrt{21} \end{bmatrix}$$

Solve by (back) substitution...

Write equations in algebraic form:

$$\sqrt{14}\, x_1 + \frac{10}{\sqrt{14}} x_2 = \frac{1}{\sqrt{14}}$$

$$\frac{\cancel{4}}{7}\sqrt{21}\, x_2 = \frac{\cancel{4}}{\sqrt{21}} \qquad x_2 = \frac{7}{21} = \frac{1}{3}$$

mult by $\sqrt{14}$

$$14x_1 + 10x_2 = 1 \qquad x_1 = \frac{1 - 10x_3}{14} = \frac{-7/3}{14} = \frac{-1}{6}$$

The vector x which minimizes $\|Ax - b\|$ is

$$x = \begin{bmatrix} -1/6 \\ 1/3 \end{bmatrix}$$

In this course we solve $Ax=b$ in many ways.

Gaussian elimination using row operations..

triangular

① Factor $A=LU$ and then solve the triangular system of systems

$LUx=b$

$y=Ux$

$Ly=b$

$$\begin{cases} Ly=b \\ Ux=y \end{cases}$$

Solve by substitution.

$Q^TQ=I$

triangular

② Factor $A=QR$ and then solve

Gram-Schmidt using column operations

$$Rx=Q^Tb$$

solve by substitution

③ Using Cramer's rule...

$$x_i = \frac{\det A_i(b)}{\det A}$$

note the way to find the determinant is using Gauss elimination...

$\det A = \det U$

when $A=LU$.

least squares solution of $A\mathbf{x} = \mathbf{b}$.

15. $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$

In Exercises 17–26, A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$. Mark

for next time work one of these...