15. $A = \begin{bmatrix}2 & 3\\ 2 & 4\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2/3 & -1/3\\ 2/3 & 2/3\\ 1/3 & -2/3\end{bmatrix}\begin{bmatrix}3 & 5\\ 0 & 1\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}7\\ 3\\ 1\end{bmatrix}$

16. $A = \underbrace{\begin{bmatrix}1 & -1\\ 1 & 4\\ 1 & -1\\ 1 & 4\end{bmatrix}}_{\text{data}} = \begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\\ 1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}2 & 3\\ 0 & 5\end{bmatrix}$, $\mathbf{b} = \underbrace{\begin{bmatrix}-1\\ 6\\ 5\\ 7\end{bmatrix}}_{\cancel{\text{data}}}$

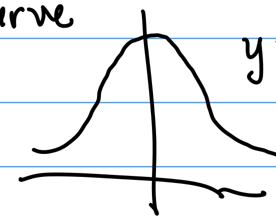
$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

parameters in model being fitted...

\#15 Find the x that minimizes $\|Ax - b\|$

minimizing $\|Ax-b\|$ is related to fitting statistical models because of the square in the normal distribution is the same square in the norm (after some statistics).

bell curve



$y = e^{-x^2}$

$A = \begin{bmatrix}2 & 3\\ 2 & 4\\ 1 & 1\end{bmatrix} = \overbrace{\begin{bmatrix}2/3 & -1/3\\ 2/3 & 2/3\\ 1/3 & -2/3\end{bmatrix}}^{Q}\overbrace{\begin{bmatrix}3 & 5\\ 0 & 1\end{bmatrix}}^{R}$, $\mathbf{b} = \begin{bmatrix}7\\ 3\\ 1\end{bmatrix}$

To minimize $\|Ax-b\|$ we could solve

① $A^TAx = A^Tb$ (normal equations)

② $Rx = Q^Tb$ (using $A = QR$).

$$Q^T b = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Solve $Rx = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

in standard algebraic form

$$3x_1 + 5x_2 = 7$$
$$x_2 = -1$$

$x_2 = -1$

$x_1 = \dfrac{7 - 5x_2}{3} = 4$

Therefore the minimizer is $\boxed{x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}}$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

Try to minimize $\|Ax - b\|$. Solve $Rx = Q^T b$

$$Q^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

$$\begin{array}{r} {}^{3}17 \\ \times\ 5 \\ \hline 85 \end{array} \qquad \begin{array}{r} 85 \\ -27 \\ \hline 58 \end{array}$$

Solve by (back) substitution...

$$\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

$$\frac{17}{2} - \frac{27}{10} = \frac{85 - 27}{10} = \frac{58}{10}$$
$$= \frac{29}{5}$$

$$2x_1 + 3x_2 = 17/2$$
$$5x_2 = 9/2$$

$x_2 = \dfrac{9}{10}$

$x_1 = \dfrac{17/2 - 3x_2}{2} = \dfrac{29}{10}$

Therefore the minimizer is $x = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$.

chapter 5

Next topic: Eigenvalues and eigenvectors...

Note: ① Solving for eigenvalues and eigenvectors involves solving a quadratic equation in many variables... (hard) Bad

② Converts matrix-vector multiplication into scalar-vector multiplication (which is easier). Good

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

$$\boxed{\underbrace{Ax}_{\text{matrix-vector multiplication}} = \underbrace{\lambda x}_{\text{scalar-vector multiplication...}}}$$

Given the matrix A how to solve for x and λ. How to find the eigenvectors and the eigenvalues...

Let $A \in \mathbb{R}^{n \times n}$

only works for square matrices

$x \in \mathbb{R}^n$

$\lambda \in \mathbb{R}$

since $Ax = \lambda x$ implies the dimension of the input for the linear function A is the same as the output.

use determinants to solve for λ, then its just a linear equation in x...

Solve for x and λ — total of $n+1$ unknowns.

$$Ax = \lambda x$$

quadratic in the unknowns...

How many equations? n equations...

When more unknowns than equations, the problem is underdetermined and likely has infinite number of solutions. (but maybe not).

Note is λ, x are a solution

$$3Ax = 3\lambda x$$

$$A(3x) = \lambda(3x)$$

Then λ and $3x$ are also a solution...

So if there are any solutions there are an infinite number by rescaling x.