Eigenvalue - Eigenvector problem

$$Ax = \lambda x$$ solve for λ and x.

Note that the length of x is not determined by this equation

One could insist that x be a unit vector to (partially) remedy this problem.

In some cases taking $\|x\| = 1$ is useful.

How to solve this problem: Use a computer... the (good) algorithms work by iterative approximations and that's not easy to do by hand...

Instead we use a theoretical approach based on determinants that work for 3x3 matrices...

How?

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$\boxed{(A - \lambda I)x = 0}$$

use determinants to eliminate x so we can solve for λ

Looking for λ and x that satisfies the eigenvalue eigenvector problem. Namely, $x \neq 0$, This equation says that $x \in \text{Nul}(A-\lambda I)$.

Since $x \neq 0$ we ask what values of λ are such that this nullspace is nontrivial?

Answer: can tell with determinants

In particular $\text{Nul}(A-\lambda I)$ is non-trivial means that $A-\lambda I$ is missing some pivots, and thus means $A-\lambda I$ is not invertible, thus $\det(A-\lambda I)=0$.

Strategy for solving $Ax=\lambda I$.

① Solve for λ such that $\det(A-\lambda I)=0$

② For each such λ find $x \in \text{Nul}(A-\lambda I)$ such that $x \neq 0$.

Example:

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

Question: find the eigenvalues (and eigenvectors).

§5.2 #7

$A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ ① Solve for λ such that $\det(A-\lambda I)=0$

② For each such λ find $x \in \text{Nul}(A-\lambda I)$ such that $x \neq 0$.

$$A-\lambda I=\begin{bmatrix}5&3\\-4&4\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}5-\lambda&3\\-4&4-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda&3\\-4&4-\lambda\end{bmatrix}=(5-\lambda)(4-\lambda)-(3)(-4)$$

$$=\lambda^2-9\lambda+20+12=\underbrace{\lambda^2-9\lambda+32}_{\text{characteristic polynomial of } A}=0$$

$a=1,\ b=-9,\ c=32$

characteristic polynomial of A

$\chi(\lambda)=\det(A-\lambda I)$

$$\begin{array}{r}32\\ \times\ 4\\ \hline 128\\ -81\\ \hline 47\end{array}$$

Don't see easy factors... was there a mistake? If not, use quadratic formula to solve for λ.

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{9\pm\sqrt{81-128}}{2}=\frac{9\pm\sqrt{-47}}{2}$$

Answer: the eigenvalues are $\frac{9+i\sqrt{47}}{2}$ and $\frac{9-i\sqrt{47}}{2}$.

Note: if $A=A^t$ you never get complex eigenvalues, but otherwise you might or might not

Since solving for the $\text{Nul}(A-\lambda I)$ is going to involve complex arithmetic, let's just do the next problem... To do this it's just Gaussian elimination with row operations.

§5.2 #9

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$$

① Solve for λ such that $\det(A-\lambda I)=0$

② For each such λ find $x \in \mathrm{Nul}(A-\lambda I)$ such that $x \neq 0$.

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & 0-\lambda \end{bmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} 3-\lambda & -1 \\ 6 & 0-\lambda \end{bmatrix} - 2\det\begin{bmatrix} 0 & -1 \\ 6 & 0-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(3-\lambda)(-\lambda)+6\right] - 2\cdot 6$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 6) - 12$$

$$= \lambda^2 - 3\lambda + 6 - \lambda^3 + 3\lambda^2 - 6\lambda - 12$$

$$= -\lambda^3 + 4\lambda^2 - 9\lambda - 6 \overset{set}{=} 0.$$

Solve $\lambda^3 - 4\lambda^2 + 9\lambda + 6 = 0$

factors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$

plug in $\lambda = 1$

$$1^3 - 4\cdot 1^2 + 9\cdot 1 + 6 \neq 0$$

plug in $\lambda = -1$

$$-1 - 4 - 9 + 6 \neq 0$$

plug in $\lambda = -2$

$$-8 - 16 - 18 + 6 \neq 0$$

$\lambda = 2$

$$8 - 16 + 18 + 6 \neq 0$$

Hard to find the eigenvalues, but...

The characteristic polynomial is

$$\chi(\lambda) = -\lambda^3 + 4\lambda^2 - 9\lambda - 6$$

and that's all the problem really asked...