Chapter 5.3

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

Solve the eigenvalue eigenvector problem $Ax = \lambda x$ when

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

Strategy for solving $Ax = \lambda I$.

1. Solve for λ such that $\det(A - \lambda I) = 0$
2. For each such λ find $x \in \text{Nul}(A - \lambda I)$ such that $x \neq 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix} = (3-\lambda)(5-\lambda) + 1$$

$$= \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$$

$\lambda = 4$ with mult 2.

note usualy there are two eigenvalue...

Now solve for x in $\text{Nul}(A - \lambda I)$

$\lambda = 4$

$$\begin{bmatrix} 3-4 & -1 \\ 1 & 5-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} P & F \end{array}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 = 0$

$\boxed{x_1 = -x_2}$

free vbl.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$

eigenvector

Thus $\lambda=4$ is an eigenvalue for eigenvector $x=\begin{bmatrix}-1\\1\end{bmatrix}$.

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13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

$$\chi(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{bmatrix}$$

$$12 \\ +2 \\ +2 \\ \overline{16} \\ -3 \\ \overline{13} \\ -8 \\ \overline{5}$$

$$= (2-\lambda)(3-\lambda)(2-\lambda) + 2 + 2 - (3-\lambda) - 2(2-\lambda) - 2(2-\lambda)$$

$$= -\lambda^3 + 7\lambda^2 + (6+6+4)(-\lambda) + 12 + 2 + 2 - (3-\lambda) - 2(2-\lambda) - 2(2-\lambda)$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = -(\lambda-1)(\lambda^2-6\lambda+5) = -(\lambda-1)(\lambda-5)(\lambda-1)$$

Two factors for $\lambda=1$

$$\chi(1) = -1+7-11+5 = 0$$

$$\begin{array}{r} \lambda^2 - 6\lambda + 5 \\ \lambda - 1 \overline{)\ \lambda^3 - 7\lambda^2 + 11\lambda - 5} \\ -(\lambda^3 - \lambda^2) \\ \hline -6\lambda^2 + 11\lambda - 5 \\ -6\lambda^2 + 6\lambda \\ \hline 5\lambda - 5 \\ 5\lambda - 5 \\ \hline 0 \end{array}$$

Therefore the eigenvalues are 1 with mult 2 and 5 with mult 1.

$$-16 \\ +1 \\ +2 \\ 2 \\ \overline{-11}$$

Now find the eigenvectors... $\mathrm{Nul}(A-\lambda I)$

$\mathrm{Nul}(A-\lambda I)$

$\lambda = 1$

$$A-\lambda I = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} P & F & F \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r_2 \leftarrow r_2 - r_1$

$r_3 \leftarrow r_3 + r_1$

become zero . .

$x_1 + 2x_2 - x_3 = 0$

$x_1 = -2x_2 + x_3$

Free vbls

Free vbl.

Solution:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$$

Basis for $\mathrm{Nul}(A-\lambda I)$

$$\mathrm{Nul}(A-\lambda I) = \mathrm{Col}\left(\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Therefore $\lambda = 1$ is an eigenvalue with two linearly independent eigenvectors $x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and also $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Note these eigenvectors are not unique.

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{where not } c_1 = c_2 = 0$$

this is also an eigenvector with $\lambda = 1$.

Check this $\quad Ax = \lambda x$

$$\underbrace{\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}}_{A} \Bigg(\underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{x} \Bigg) = \underbrace{1}_{\lambda} \Bigg(\underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{x} \Bigg)$$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 5$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix}$$

$\boxed{\text{Find Nul } A - \lambda I}$

$$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -3 & 2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \qquad \begin{array}{l} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 + r_1 \end{array}$$

$$\begin{array}{ccc} P & P & F \\ \end{array}$$
$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix} \qquad \begin{array}{l} r_3 \leftarrow r_3 - r_2 \\ r_2 \leftarrow \frac{1}{-4} r_2 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 + 2r_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0 \qquad x_1 = -x_3$

$x_2 + x_3 = 0 \qquad x_2 = -x_3$

Solution $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

eigenvector for $\lambda = 5$