§7.1

Exercises

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ Find eigenvalues and eigenvectors of this matrix

14. $\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$ Find eigenvalues and eigenvectors of this matrix

Trying to solve $Ax = \lambda x$ for λ and x. doesn't determine the length of x...

① Solve $\det(A - \lambda I) = 0$ for λ

② Substitute λ back in and find x to such that $x \in \mathrm{Nul}(A - \lambda I)$

$A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$

① $\det \begin{bmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 25 = \lambda^2 - 2\lambda - 24$

$= (\lambda - 6)(\lambda + 4) = 0$

So eigenvalues are $\lambda = 6$ and $\lambda = -4$.

- Note the values of λ in this case were real. This always happens when $A^T = A$.

Why?

② Now find $x \in \mathrm{Nul}(A - \lambda I)$ for $\lambda = 6$ and $\lambda = -4$.

$\lambda = 6$

$$\begin{bmatrix}1 & -5\\ -5 & 1\end{bmatrix} - 6\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{array}{c}\begin{matrix}P & \;\;F\end{matrix}\\ \begin{bmatrix}-5 & -5\\ -5 & -5\end{bmatrix}\end{array}$$

Note since we solved for λ such that $\det(A-\lambda I)=0$ then the Nullspace of $A-\lambda I$ must be non-trivial. There must be at least one free vbls.

Since it's 2×2 matrix that leaves only one equation to find x:

$$-5x_1 - 5x_2 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_2\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix}x_2$$

eigenvector.

$\lambda = -4$

$$\begin{bmatrix}1 & -5\\ -5 & 1\end{bmatrix} - (-4)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{array}{c}\begin{matrix}P & \;\;F\end{matrix}\\ \begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix}\end{array}$$

$$5x_1 - 5x_2 = 0$$
$$x_1 = x_2$$

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_2\\ x_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}x_2$$

other eigenvector.

Now factor $A = SDS^{-1}$.

$$S = \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix} \qquad D = \begin{bmatrix}6 & 0\\ 0 & -4\end{bmatrix}$$

Note the columns of S are vectors that are perpendicular to each other. This always happens when $A = A^T$.

Verify this $\begin{bmatrix}-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix} = -1+1=0$ ☑ perpendicular

Turn eigen vectors into unit vectors...

$\lambda=6$ $\quad \dfrac{\begin{bmatrix}-1\\1\end{bmatrix}}{\left\|\begin{bmatrix}-1\\1\end{bmatrix}\right\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

unit eigenvectors

$\lambda=-4$ $\quad \dfrac{\begin{bmatrix}1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

columns of P are unit eigenvectors of a symmetric matrix... $A=A^T$.

$P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$P^T = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

Multiply P^TP

$P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$P^T = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Note since the column of P are orthonormal vectors then $P^TP=I$. Since P is square then $P^{-1}=P^T$.

Now factor $A = PDP^{-1} = PDP^T$ ← don't need to compute P^{-1} when A is symmetric.

Check that this factorization worked ... $\boxed{\checkmark}$ okay

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -6/\sqrt{2} & -4/\sqrt{2} \\ 6/\sqrt{2} & -4/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

Two things: If $A = A^T$ then

1. the eigenvalues λ are real
2. the eigenvectors can be chosen so they are orthonormal.

Save the 3x3 example for next time...

For now, why are ① and ② consequences of $A = A^T$.

② why are eigenvectors perpendicular...

$$A x_1 = \lambda_1 x_1 \qquad A x_2 = \lambda_2 x_2$$

and suppose $\lambda_1 \neq \lambda_2$.

Claim $x_1 \cdot x_2 = 0$.

since A is symmetric.

$$\lambda_1 x_1 \cdot x_2 = A x_1 \cdot x_2 = (A x_1)^T x_2 = x_1^T A^T x_2 = x_1^T A x_2$$

$$= x_1 \cdot A x_2 = x_1 \cdot \lambda_2 x_2 = \lambda_2 x_1 \cdot x_2$$

Thus $\lambda_1 x_1 \cdot x_2 = \lambda_2 x_1 \cdot x_2$

$$(\lambda_1 - \lambda_2) x_1 \cdot x_2 = 0$$

since $\lambda_1 \neq \lambda_2$ then $\lambda_1 - \lambda_2 \neq 0$ so $x_1 \cdot x_2 = 0$.

① Why are the eigenvalues real?

$$Ax = \lambda x$$

complex conjugate ...

real means that $\bar{\lambda} = \lambda$

what is complex conjugate?

$$\overline{3+4i} = 3-4i \neq 3+4i$$

If the imaginary part is non-zero these are two different numbers.

$$\overline{\lambda x} = \overline{Ax} = \underbrace{\bar{A}\,\bar{x}}_{\text{since } A \text{ is a real matrix}} = A\bar{x}$$

$$\bar{A} = \overline{\begin{bmatrix} 2+i & 3-4i \\ 6+i & 7+2i \end{bmatrix}} = \begin{bmatrix} \overline{2+i} & \overline{3-4i} \\ \overline{6+i} & \overline{7+2i} \end{bmatrix} = \begin{bmatrix} 2-i & 3+4i \\ 6-i & 7-2i \end{bmatrix}$$

why is that a reasonable definition of $\bar{A}$?