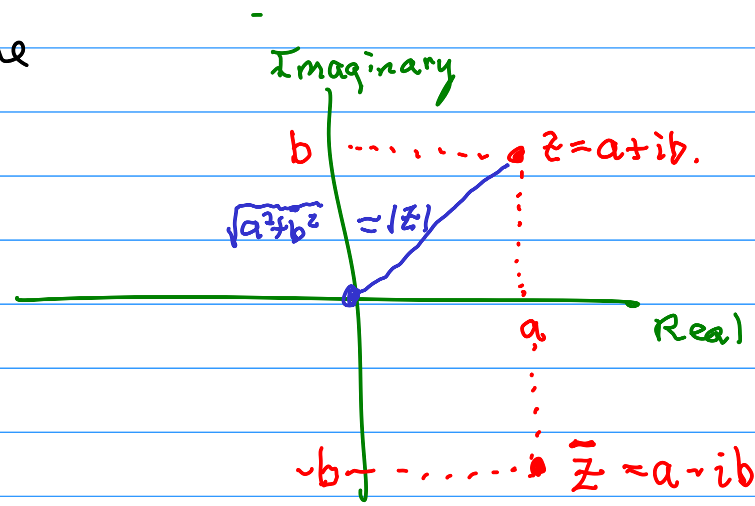
Complex numbers

$z = a + ib$ where $a, b \in \mathbb{R}$

Complex plane



Complex conjugate

$\bar{z} = a - ib$ (swap i for $-i$ everywhere)

Norm (modulus or absolute value) of a complex number

$|z| = \sqrt{a^2+b^2}$

Note, can express the modulus in term of complex conjugates

$i^2 = -1$

$$z\bar{z} = (a+ib)(a-ib) = a^2 - \cancel{iab} + \cancel{iab} - i^2 b^2$$

$$z\bar{z} = a^2 + b^2 = |z|^2$$

$$|z| = \sqrt{z\bar{z}}$$

Example: $z = 2+3i \qquad w = 4+5i$

$i^2 = -1$

$$zw = (2+3i)(4+5i) = 8 + 10i + 12i + 15i^2$$

$$= 8 + 22i - 15 = -7 + 22i$$

$$\overline{zw} = \boxed{-7 - 22i}$$

conjugation means swap i with $-i$ everywhere...

Now conjugate first and then multiply

$i^2 = -1$

$$\bar{z}\,\bar{w} = (2-3i)(4-5i) = 8 - 10i - 12i + 15i^2$$

$$= 8 - 22i - 15 = \boxed{-7 - 22i}$$

Conclusion: $\overline{zw} = \bar{z}\,\bar{w}$.

This is not so surprising because both i and $-i$ are square roots of -1, so the same algebra works with either...

Vectors of complex numbers $x \in \mathbb{C}^n$

n dimensional vector of complex numbers

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix}$$

Question: what is $\|x\|$?

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

$$= \sqrt{a_1^2+b_1^2 + a_2^2+b_2^2 + \dots + a_n^2+b_n^2}$$

$$= \sqrt{x_1\bar{x}_1 + x_2\bar{x}_2 + \dots + x_n\bar{x}_n}$$

$$= \sqrt{\boxed{x \cdot \bar{x}}}$$

Note $x \cdot \bar{x}$ is always real even if x is a complex vector

where $\bar{x} = \begin{bmatrix} a_1 - ib_1 \\ a_2 - ib_2 \\ \vdots \\ a_n - ib_2 \end{bmatrix}$

This motivates why $\bar{x}$ of a vector means to conjugate the elements of the vector...

Another reason that $\bar{x}$ is the conjugate of each element individually is that conjugation isn't so much a function but the subtitution of $-i$ for i everywhere..

Spectral Theorem for symmetric real-valued Matrices

Two things: If $A = A^T$ then $A \in \mathbb{R}^{n\times n}$

① the eigenvalues λ are real $\square$

② the eigenvectors can be chosen so they form ~~are~~ an orthonormal. basis $\mathbb{R}^n$.

$\boxed{?}$

? not do this, but it will be done in 430 lin. alg II.

① Why does $A^T = A$ imply the eigenvalues are real?

Consider $Ax = \lambda x$. Then.

$$x \cdot \bar{x} = \|x\|^2 > 0 \text{ since } x \neq 0.$$

Now,

$$\lambda x \cdot \bar{x} = Ax \cdot \bar{x} = (Ax)^T \bar{x} = x^T A^T \bar{x}$$

$$= \underbrace{x \cdot A^T \bar{x} = x \cdot A \bar{x}}_{\text{symmetric}} = x \cdot \underbrace{\bar{A}}_{\text{since } A \in \mathbb{R}^{n \times n}} \bar{x}$$

$$= x \cdot \overline{Ax} = x \cdot \overline{\lambda x} = x \cdot \bar{\lambda} \bar{x} = \bar{\lambda}\, x \cdot \bar{x}$$

Therefore

$$\lambda \cancel{x \cdot \bar{x}} = \bar{\lambda} \cancel{x \cdot \bar{x}} \quad \text{and} \quad x \cdot \bar{x} = \|x\|^2 > 0$$

Then $\lambda = \bar{\lambda}$ means λ is real.

If $A = A^T$ and $A \in \mathbb{R}^{n \times n}$ then can factor A as

$$A = PDP^T$$

Diagonal matrix with real numbers on the diagonal.

matrix with orthonormal columns so that $P^{-1} = P^T$

Even if A is not symmetric it would be nice to do something with the spectral theorem...

$$B = A^T A$$

Then

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$

Singular value decomposition (Monday) relates the factorization

$$B = PDP^T$$

Back to say something about A.