Singular Value decomposition:

A is an arbitrary matrix, $B = A^T A$ then $B = B^T$. Then we can use the spectral theorem...

$$B = UDU^T$$

diagonal matrix with eigenvalues of B.

an orthogonal matrix made of the (unit) eigenvectors of B.

$$U = \begin{bmatrix} x_1 \,|\, x_2 \,|\, \cdots \,|\, x_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

Since $U^T = U^{-1}$ then $x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Want to say something about A (need to untangle it from the A^T).

What is Ax_i? $y_i = Ax_i$ How do the eigenvectors of B interact with A?

$$y_i \cdot y_j = Ax_i \cdot Ax_j = (Ax_i)^T A x_j = x_i^T A^T A x_j = x_i \cdot Bx_j$$

since x_j is an eigenvector of B

$$= x_i \cdot \lambda_j x_j = \lambda_j \, x_i \cdot x_j = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note when $i = j$ then $y_i \cdot y_i = \lambda_i$ thus $\|y_i\| = \sqrt{\lambda_i}$

The lengths of the y_i are the square roots of the eigenvalues of B.

$$z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}}$$ in other words $\boxed{y_i = \sqrt{\lambda_i}\, z_i}$

divided by $\sqrt{\lambda_i}$ assumes $\lambda_i \neq 0$.

Put the z_i's into a matrix

$$V = \begin{bmatrix} z_1 \,|\, z_2 \,|\, \cdots \,|\, z_n \end{bmatrix}$$ note $V^T V = I$ or $V^T = V^{-1}$.

an orthogonal matrix made of the (unit) vectors z_i which are perpendicular to each other.

Preserve angles and lengths → orthogonal matrix

diagonal matrix stretch along the coordinate axis...

Singular Value decomposition: $A = V \Sigma U^T$ ← another orthogonal matrix

where $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots & \sqrt{\lambda_n} \end{bmatrix}$ and U and V are as before.

Check the factorization:

$$AU = A\begin{bmatrix} x_1 | x_2 | \cdots | x_n \end{bmatrix} = \begin{bmatrix} Ax_1 | Ax_2 | \cdots | Ax_n \end{bmatrix}$$

by definition of the y_i's

$$= \begin{bmatrix} y_1 | y_2 | \cdots | y_n \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} z_1 | \sqrt{\lambda_2} z_2 | \cdots | \sqrt{\lambda_n} z_n \end{bmatrix}$$

note if $\lambda_i = 0$ then it doesn't matter what the corresponding z_i is chosen...

since $y_i = \sqrt{\lambda_i} z_i$

$$= \begin{bmatrix} z_1 | z_2 | \cdots | z_n \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots & \sqrt{\lambda_n} \end{bmatrix} = V\Sigma$$

Thus $AU = V\Sigma$ or $\boxed{A = V \Sigma U^T}$ since $U^{-1} = U^T$.

singular value decomposition of A.

Example: $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ square but not symmetric...

$$B = A^T A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\det(B - \lambda I) = \det \begin{bmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (8-\lambda)(5-\lambda) - 4$$

$= \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$

eigenvalues of B are 4 and 9.

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Now need the eigen vectors of B

$\lambda = 4$

$$\text{Nul}(B - 4I) = \text{Nul}\begin{pmatrix} 8-4 & 2 \\ 2 & 5-4 \end{pmatrix} = \text{Nul}\begin{array}{cc} P & F \\ \end{array}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$2x_1 + x_2 = 0$

$x_1 = -\dfrac{x_2}{2}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2$$

eigenvector for $\lambda = 4$

(Before making it into a unit eigenvector do $\lambda = 9$)

$\lambda = 9$

$$\text{Nul}\begin{bmatrix} 8-9 & 2 \\ 2 & 5-9 \end{bmatrix} = \text{Nul}\begin{array}{cc} P & F \\ \end{array}\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

$-1x_1 + 2x_2 = 0$

$x_1 = 2x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2$$

eigenvecto for $\lambda = 9$.

For U need the unit eigenvectors

for $\lambda = 4$

$$\frac{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\|} \frac{2}{2} = \frac{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$\lambda = 9$

$$\frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Thus

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Finally, what is V? The unit vectors made out of Ax_i where x_i is an eigenvector of B.

$$z_1 = \frac{A\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\left\| A\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} -4 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\|} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$z_2 = \frac{A\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{A\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\|} \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$V = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

The singular value decomposition: $A = V\Sigma U^T$.

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

That was a lot of work since one needed to find $B = A^TA$ and then the eigenvectors of B first. On the exam the eigenvectors and eigenvalues for B are provided and all that's left is to plug them into the matrices U, Σ and V.

For example, the last question on the second sample final reads as

13. Let $B = A^TA$ where A is given by

$$A = \begin{bmatrix} 0 & -5/2 \\ 2 & -3/2 \end{bmatrix}.$$

Note that B has eigenvalues and eigenvectors given by

$$\lambda_1 = 10, \quad x_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 5/2, \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the singular value decomposition $A = V\Sigma U^T$ where Σ is a diagonal matrix and U and V are orthogonal matrices.

From this one knows that

$$U = \left[\frac{x_1}{\|x_1\|} \;\middle|\; \frac{x_2}{\|x_2\|}\right], \quad \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \text{ and } V = \left[\frac{Ax_1}{\|Ax_1\|} \;\middle|\; \frac{Ax_2}{\|Ax_2\|}\right]$$

Specifically

$$\frac{x_1}{\|x_1\|} = \frac{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\left\|\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\frac{x_2}{\|x_2\|} = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\left\|\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

implies $\quad U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

and

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{5/2} \end{bmatrix}$$

while

$$Ax_1 = \begin{bmatrix} 0 & -5/2 \\ 2 & -3/2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \quad \text{so} \quad \frac{Ax_1}{\|Ax_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 0 & -5/2 \\ 2 & -3/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \end{bmatrix} \quad \text{so} \quad \frac{Ax_2}{\|Ax_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

implies $\quad V = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$