$\lambda = 1$

$$= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$$

Basis of Nul$(A - 1I)$
when $\lambda = 1$.

$\lambda = 5$

$$= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

Basis of Nul$(A - 5I)$

**13.** $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} = A$

eigenvalues and eigenvectors of A

| $\lambda$ | $x$ | |
|---|---|---|
| 1 | $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ | $A\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ |
| 1 | $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ | $A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ | $A\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ |

Lets put eigenvectors into the matrix S

$$S = \left[ x_1 \,\middle|\, x_2 \,\middle|\, x_3 \right] = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 5\cdot(-1) \\ 1 & 0 & 5(-1) \\ 0 & 1 & 5(1) \end{bmatrix}$$

$$AS = A\left[ x_1 \,\middle|\, x_2 \,\middle|\, x_3 \right] = \left[ Ax_1 \,\middle|\, Ax_2 \,\middle|\, Ax_3 \right]$$

$$= \left[ \lambda_1 x_1 \,\middle|\, \lambda_2 x_2 \,\middle|\, \lambda_3 x_3 \right] = \left[ x_1 \,\middle|\, x_2 \,\middle|\, x_3 \right] \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

Thus,

$$AS = SD$$

If S is invertible we can write this as

$$A = \underbrace{SDS^{-1}}_{\text{factorization of } A \ldots}$$

# Other matrix factorizations

$A = LU$ — Gaussian elimination using row operations

both matrices are triangular, so this made it easy to solve $Ax = b$. How?

$$\begin{cases} Ly = b \\ Ux = y \end{cases}$$ by substitution.

$A = QR$ — Gram-Schmidt orthogonalization, column operations

upper triangular

matrix had orthonormal columns so $Q^TQ = I$

Solve $Ax = b$. How: $QRx = b$ mult by $Q^T$

$$\{ Rx = Q^Tb$$ by substitution

(Another advantage, the solution of $Rx = Q^Tb$ was the minimizer of $\|Ax - b\|$ when $Ax = b$ is overdetermined and doesn't have a solution.)

not special, just another matrix.

$A = SDS^{-1}$

diagonal matrix, both upper and lower triangular. very simple

Though S is just a bunch of numbers, the way it appears in the factorization is special.

Obtained by solving the eigenvalue eigenvector problem $Ax = \lambda x$ for $\lambda$ and $x$.

Here the eigenvectors are the columns of S and the eigenvalues the diagonal of D

$$A^2 = (SDS^{-1})(SDS^{-1}) = SD\cancel{S^{-1}}\cancel{S}DS^{-1}$$

$$A^2 = SD^2S^{-1}$$

- Because of the way $S$ appears in the factorization, the square appears only on the simple matrix $D$.

Note also

$$A^3 = SD^3S^{-1}$$
$$A^4 = SD^4S^{-1}$$
$$\vdots$$
$$A^k = SD^kS^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

What is $D^2$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 5^2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$$

In fact, (almost) any function that can be applied to the numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$ can now be applied to a matrix.

Examples

$$\sin\left(\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}\right) = S(\sin D)S^{-1}$$

$$= S\begin{bmatrix} \sin 1 & 0 & 0 \\ 0 & \sin 1 & 0 \\ 0 & 0 & \sin 5 \end{bmatrix}S^{-1}$$

$$A^{1/2} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}^{1/2} = S \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} S^{-1}$$

Check that it works...

$$A^{1/2}A^{1/2} = S \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \cancel{S^{-1}} \cancel{S} \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} S^{-1}$$

$$= S \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} S^{-1}$$

$$= SDS^{-1} = A.$$

Possible problems: what if the diagonal entries are negative?

General problem: what is $S$ is not invertible?

in this case the factorization didn't even exist...

1. $S$ must be square, so a matrix $A \in \mathbb{R}^{n \times n}$ must have $n$ eigenvectors.

2. Those eigenvectors must be linearly independent.

Recall the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

from last time had only one linearly independent eigenvector $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 4$.

Thus, $A$ can not be factored as $A = SDS^{-1}$ where $D$ is a diagonal matrix. If you could write $A$ as $SDS^{-1}$ that would imply the columns of $S$ were eigenvectors of $A$ and the diagonal entries of $D$ the corresponding eigenvalues, which couldn't be the case because there is only one linearly independent eigenvector.

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Note that if $x_1$ and $x_2$ are eigenvectors for different eigenvalues $\lambda_1 \neq \lambda_2$. Then $x_1$ and $x_2$ must be linearly independent.

Thus, if you have enough eigenvectors, you already know $S$ is invertible.