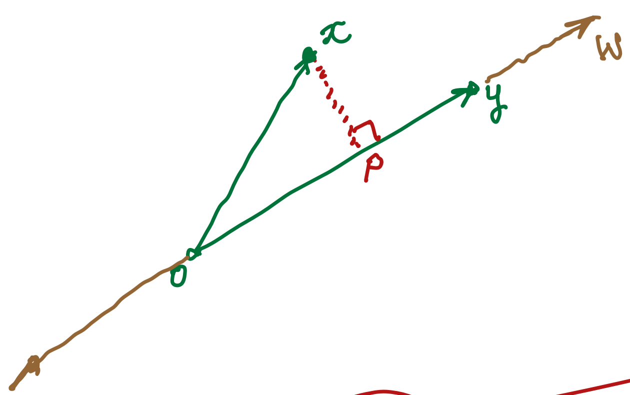


$q = \frac{y}{\|y\|}$ $\quad \boxed{p = (q \cdot x) q}$

$W = \text{span}\{y\} = \text{span}\{q\}$

write $x = p + z$

where $p \in W$ and $z \in W^{\perp}$

Generalize this

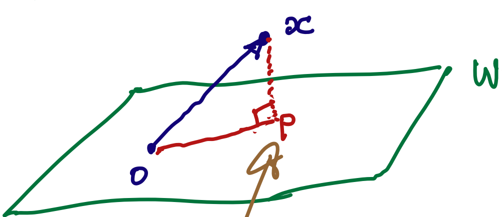
Suppose instead of 1 dimensional that W is higher dimensional subspace.

$W = \text{span}\{v_1, v_2, \ldots, v_n\} = \text{Col } A$

set of vectors $\{v_1, v_2, \ldots, v_n\}$

Example W is 2 dimensional

$A = \begin{bmatrix} v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n \end{bmatrix}$



geometrically not that p is also the closest point in W to x.

useful in optimization problem.

if the vectors are independent then this is a basis and $\dim W = n$.

Question: How to find p?

I'm looking for something similar to $p = (q \cdot x) q$ for higher dimensional subspaces.

Algorithm: Gram-Schmidt algorithm:

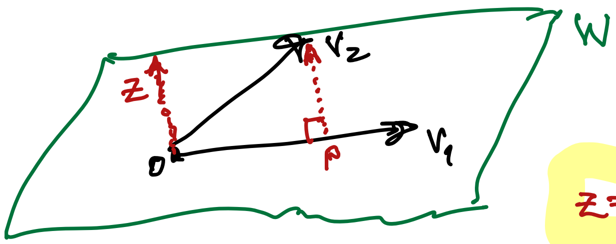
1. Column operations to factor a matrix $A = QR$ where $Q^T Q = I$ and R is upper triangular.

2. Is a way to form a basis for W

into the higher dimensional version of the q in $p=(q\cdot x)q$.

Suppose $W=\text{span}\{v_1,v_2\}$ and that these vectors are independent. Thus $\{v_1,v_2\}$ form a basis of W and $\dim W=2$.

$v_1,v_2\in W$



$p=(q\cdot v_2)q$ where $q=\frac{v_1}{\|v_1\|}$

$z=v_2-p$

Note that $W=\text{span}\{v_1,z\}$

Trying to explain why this is the same...

$v_2=z+p$

Why? $W=\text{span}\{v_1,v_2\}=\text{Col}\,A$ where $A=[v_1|v_2]$. Thus any point $w\in W$ can be written $w=Ax$ for some $x\in\mathbb{R}^2$. Thus...

$$w=x_1v_1+x_2v_2$$

$$w=x_1v_1+x_2(z+p)$$

$$w=x_1v_1+x_2(z+(q\cdot v_2)q)$$

$$=x_1v_1+x_2z+x_2(q\cdot v_2)\frac{v_1}{\|v_1\|}$$

$$w=\left(x_1+\frac{x_2(q\cdot v_2)}{\|v_1\|}\right)v_1+x_2z$$

This shows that $w\in W$ can be written as a linear combination of v_1 and z. So $w\in\text{Span}\{v_1,z\}$

So $W=\text{Span}\{v_1,v_2\}=\text{span}\{v_1,z\}=\text{span}\{q_1,z\}$

where $z=v_2-p$

$p=(q_1\cdot v_2)q_1$ and $q_1=\frac{v_1}{\|v_1\|}$

Note here we are subtracting something from v_2 so it's perpendicular to v_1.

Now define $q_2 = \frac{z}{\|z\|}$ then $W = \text{span}\{q_1, q_2\}$

Gram-Schmidt algorithm generalizes this idea of subtracting stuff to make vectors perpendicular to larger sets of vectors...

Let $\{v_1, v_2, \ldots, v_d\}$ be a basis for W. Then

$z_1 = v_1$ $\qquad q_1 = \frac{z_1}{\|z_1\|}$

$z_2 = v_2 - (q_1 \cdot v_2) q_1$ $\qquad q_2 = \frac{z_2}{\|z_2\|}$

$z_3 = v_3 - (q_1 \cdot v_3) q_1 - (q_2 \cdot v_3) q_2$ $\qquad q_3 = \frac{z_3}{\|z_3\|}$

$\vdots$ $\qquad \vdots$

$z_n = v_n - (q_1 \cdot v_n) q_1 - (q_2 \cdot v_n) q_2 - \cdots - (q_{n-1} \cdot v_n) q_{n-1}$

$q_n = \frac{z_n}{\|z_n\|}$

When done

- $q_1 \cdot q_2 = 0$ $\quad q_1 \cdot q_3 = 0$ $\cdots$ $q_i \cdot q_j = 0$ for $i \neq j$

  that is all the q_i's are perpendicular to each other.
- Also $\|q_i\| = 1$ for all vectors q_i.
- $W = \text{span}\{q_1, q_2, \ldots, q_n\}$

Next time use this to factor $A = QR$