Math 330: Final Exam Version B Sample Final

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Write down the augmented matrix $\left[\, A \,|\, b \,\right]$ corresponding to the system of linear equations given by

$$\begin{cases} 3x_1 + x_2 - 5x_4 = 3 \\ x_2 - 3x_3 + 7x_4 = -2 \\ -4x_1 + 2x_2 + x_4 = 9 \end{cases}$$

but *do not* solve these equations.

3. Find $\det(A)$, $\det(B)$ and $\det(AB)$ where

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 8 & 0 \\ 1 & 2 & 17 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

4. Consider the matrix A with reduced row eschelon form R where

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 2 & \frac{2}{3} & 3 \\ 6 & 2 & 11 & \frac{19}{6} & \frac{25}{2} \\ -\frac{3}{2} & -\frac{1}{2} & -\frac{19}{2} & \frac{1}{12} & -\frac{9}{4} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & \frac{14}{9} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(i) Find a basis for Col(A).

(ii) Find a basis for Nul(A).

5. Let A be the matrix and x be the vector given by

$$A = \begin{bmatrix} 4 & -7 & -1 \\ 1 & -6 & 1 \\ -3 & -3 & 2 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Show that x is an eigenvector of A and find the eigenvalue.

6. Answer the following true false questions:

(i) If $x \in \operatorname{Nul} A$ and $y \in \operatorname{Nul} A$ then $x + y \in \operatorname{Nul} A$

(A) True

(B) False

(ii) If A is a matrix such that $A^T = A$ then A is invertible.

(A) True

(B) False

(iii) When two linear transformations are performed one after another, the combined effect is always a linear transformation.

(A) True

(B) False

(iv) $\det(A + B) = \det(A) + \det(B)$.

(A) True

(B) False

(v) Cramer's rule can only be used for invertible matrices.

(A) True

(B) False

(vi) If W is a subspace of $\mathbf{R}^n$ and v is in both W and $W^\perp$, then $v = 0$.

(A) True

(B) False

(vii) If $A = SDS^{-1}$ where then A and D both have the same eigenvalues.

(A) True

(B) False

(viii) If $A \in \mathbf{R}^{n\times n}$ is symmetric, there exists an orthonormal basis of $\mathbf{R}^n$ which consists of eigenvectors of A.

(A) True

(B) False

(ix) Every matrix $A \in \mathbf{R}^{n\times n}$ can be factored as $A = LU$ where L is lower triangular and U is upper triangular.

(A) True

(B) False

7. Suppose $A \in \mathbf{R}^{2\times 3}$ is given by

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 5 \end{bmatrix}.$$

How many free variables does the equation $Ax = 0$ have? Find all solutions to the equation $Ax = 0$.

8. Suppose $A \in \mathbf{R}^{2\times 2}$ is given by

$$A = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to factor $A = QR$ where Q is a matrix with orthonormal columns and R is upper triangular.

9. Find the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

10. The LU factorization of a matrix A is given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/3 & 3/2 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Explain how to use this factorization to solve the equation $Ax = b$ and then find the value of x corresponding to $b = (-6, 1, 12)$.

11. The QR factorization of a matrix A is given by

$$Q = \begin{bmatrix} \frac{2}{7} & \frac{-3}{\sqrt{13}} \\ -\frac{6}{7} & 0 \\ \frac{3}{7} & \frac{2}{\sqrt{13}} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 2 & -1 \\ 0 & \sqrt{13} \end{bmatrix}.$$

Explain how to use this factorization to minimize $\|Ax - b\|$ and then find the minimizing value of x corresponding to $b = (1, -1, 2)$.

12. The matrix A given by

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}$$

has eigenvalues λ_i and eigenvectors x_i given by

$$\lambda_1 = 8, \quad x_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1, \quad x_2 = \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}, \qquad \lambda_3 = -3, \quad x_3 = \begin{bmatrix} 4 \\ 11 \\ 0 \end{bmatrix}.$$

Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

(i) What is D?

(ii) What is S?

13. Let $B = A^T A$ where A is given by

$$A = \begin{bmatrix} 0 & -5/2 \\ 2 & -3/2 \end{bmatrix}.$$

Note that B has eigenvalues and eigenvectors given by

$$\lambda_1 = 10, \quad x_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = 5/2, \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the singular value decomposition $A = V \Sigma U^T$ where Σ is a diagonal matrix and U and V are orthogonal matrices.

(i) What is Σ?

(ii) What is U?

(iii) What is V?