Linearity property.

$$T(x) = \det\begin{bmatrix}1 & x_1 & 3\\ 4 & x_2 & 7\\ -1 & x_3 & 8\end{bmatrix}$$

Claim $T(x)$ is a linear fuction.

Note $x \in \mathbb{R}^3$, $T(x) \in \mathbb{R}$ so $T : \mathbb{R}^3 \to \mathbb{R}$.

Recursive definition of determinant, expand along a row or column in terms of the determinants of the minors.

expand on this column ..

$$T(x) = \det\underbrace{\begin{bmatrix}1 & x_1 & 3\\ 4 & x_2 & 7\\ -1 & x_3 & 8\end{bmatrix}}_{A} = (-1)^{1+2} x_1 \det A_{12} + (-1)^{2+2} x_2 \det A_{22} + (-1)^{3+2} x_3 \det A_{32}$$

$$B = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 7\\ -1 & 0 & 8\end{bmatrix} \qquad B_2(x) = A$$

$$= -x_1 \det\begin{bmatrix}4 & 7\\ -1 & 8\end{bmatrix} + x_2 \det\begin{bmatrix}1 & 3\\ -1 & 8\end{bmatrix} - x_3 \det\begin{bmatrix}1 & 3\\ 4 & 7\end{bmatrix}$$

$$= -39x_1 + 11x_2 + 5x_3 = \underbrace{\begin{bmatrix}-39 & 11 & 5\end{bmatrix}}_{1\times 3 \text{ matrix}} \underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}}_{3\text{-vector}}$$

$T(x)$ is a matrix vector product $\Rightarrow$ means a linear function...

In other words

$$\begin{cases} T(c\mathbf{x}) = cT(\mathbf{x}) & \text{for all scalars } c \text{ and all } \mathbf{x} \text{ in } \mathbb{R}^n \\ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) & \text{for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n \end{cases}$$

Example... find the determinant... using Gaussian elimination

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 2r_1$

$r_3 \leftarrow r_3 - 3r_1$

$r_4 \leftarrow r_4 - r_1$

elimination

$$= \det \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{bmatrix}$$

$r_4 \leftarrow r_4 - r_3$

$$= \det \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

row of zeros means matrix is not invertible so $\det A = 0$

or

$$T(x) = \det \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ x_1 & x_2 & x_3 & x_3 \end{bmatrix}$$

This is a linear function

$T(0) = 0$

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

How many determinants to find x?

n determinants in the numerator

1 in the denominator

$n+1$ determinants...

That is... Gaussian elimination $n+1$ times to solve $Ax=b$...

- Practically speaking it's better to simply use Gaussian elimination once on the augmented matrix $[A|b]$, or alternatively once to find the factorization $A=LU$ and then solve $Ly=b$ and $Ux=y$ to find x.

- Theoretically speaking, it's nice to have an equation that solves for x directly. In case you want to do something that needs an equation, such as Calculus.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_1(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\sim}{I}_2(x) = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix}$$

$$\boxed{\det I_1(x) = x_1 \cdot 1 \cdot 1 \cdot 1 = x_1}$$

$$\det I_2(x) = \det \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix} \qquad c_1 \longleftrightarrow c_2$$

$$= -\det \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} \qquad r_1 \longleftrightarrow r_2$$

$$= \det \begin{bmatrix} x_2 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} = x_2 \cdot 1 \cdot 1 \cdot 1 = x_2$$

Similarly

$\det I_3(x) = x_3$ and $\det I_4(x) = x_4$

$$A I_i(x) = A \begin{bmatrix} e_1 & e_2 & \cdots & \overset{i\text{th column}}{\overset{\downarrow}{x}} & \cdots & e_n \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\substack{\uparrow \\ \text{first} \\ \text{column} \\ \text{of } A}}{Ae_1} & \underset{\substack{\uparrow \\ \text{second} \\ \text{column} \\ \text{of } A}}{Ae_2} & \cdots & \overset{i\text{th column}}{\underset{\substack{\uparrow \\ b}}{Ax}} & \cdots & \underset{\substack{\uparrow \\ \text{last} \\ \text{column} \\ \text{of } A}}{Ae_n} \end{bmatrix} = A_i(b)$$

Therefore $\quad A I_i(x) = A_i(b)$

main idea

$$\boxed{\begin{aligned} \det A I_i(x) &= \det A_i(b) \\ \det A \det I_i(x) &= \det A_i(b) \end{aligned}}$$

$$x_i = \det I_i(x) = \frac{\det A_i(b)}{\det A}$$

A Formula for A^{-1}

Once you have a way to solve $Ax=b$, then its possible to find A^{-1} by solving

$$Ax = e_1 \quad , \quad Ax = e_2 \quad , \quad \ldots \quad Ax = e_n .$$

and then expressing

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n$$

and using the linearity...

We already did this using augmented matrices

$$[A|e_1] \qquad [A|e_2] \qquad \dots \qquad [A|e_n]$$

put it into one big augmented matrix

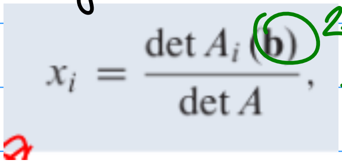
$$[A|e_1|e_2|\dots|e_n] = [A|I]$$

used Gaussian elimination to obtain the reduced echelon form of A which is I

thus $[I|A^{-1}]$

↑ That's the inverse...

Do the same thing with



$$x_i = \frac{\det A_i(\mathbf{b})}{\det A},$$

$\leftarrow e_j$

← Cramer's rule...

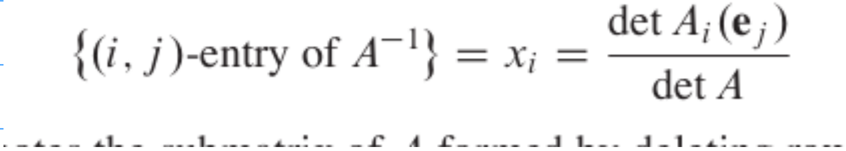
$Ax = e_1$, $x_i = \dfrac{\det A_i(e_1)}{\det A}$

$Ax = e_2$, $x_i = \dfrac{\det A_i(e_2)}{\det A}$

. . .

$Ax = e_n$, $x_i = \dfrac{\det A_i(e_n)}{\det A}$

Formula for the inverse matrix...



$$\{(i,j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

In terms of the cofactors (things that appeared in the definition of the determinant)

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji} \qquad (3)$$

where C_{ji} is a cofactor of A. By (2), the (i, j)-entry of A^{-1} is the cofactor C_{ji} divided by $\det A$. [Note that the subscripts on C_{ji} are the reverse of (i, j).] Thus

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \qquad (4)$$

transpose of how one usually indexes a matrix...