Eigenvalue-Eigenvector Problem

Given a square matrix $A \in \mathbb{R}^{n\times n}$ we look for a way to convert matrix-vector multiplication into something simpler scalar-vector multiplication

$$\underbrace{A}_{\text{matrix}}\ \underbrace{x}_{\text{vector}} = \underbrace{\lambda}_{\text{scalar}}\ \underbrace{x}_{\text{vector}}$$

Find $x \in \mathbb{R}^n$ and scalars λ that satisfy this relation

Given A find x and λ.

eigenvector

corresponding eigenvalue.

Note this is a non-linear equation because the unknown λ multiplies the other unknown x.

Solve by factoring things...

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

- note $x=0$ is always a solution to this system... However $x=0$ is not a useful solution so we ignore it...

- another possibility $A-\lambda I = 0$. However this is so unlikely that it's not going to happen...

Let $B = A - \lambda I$ then $Bx = 0$ is just finding the solutions to a homogeneous linear eq.

When does $Bx=0$ have a solution x that is not the zero solution...

If B is invertible there would be a unique solution and since $x=0$ is a solution, then that would be the only one.

If B is not invertible, then there are free variables and so $Bx=0$ has lots of solutions, most of which aren't zero.

By Cramer's rule as long at $\det B \neq 0$ then the matrix in invertible. And if B is not invertible then $\det B = 0$.

When is $\det B = 0$?

$$\det B = \underbrace{\det(A-\lambda I) = 0}_{\text{characteristic equation}}$$

condition of λ that implies there is a non-zero x...

$\chi(\lambda) = \det(A-\lambda I)$

characteristic polynomial

12. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \lambda = 1, 5$

$$\chi(\lambda) = \det(A-\lambda I) = \det\left(\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{bmatrix}\right) = (7-\lambda)(-1-\lambda) - (-3)(4)$$

$= (\lambda-7)(\lambda+1)+(3)(4) = \lambda^2-6\lambda-7+12$

$= \lambda^2-6\lambda+5$

characteristic polynomial...

Thus $\chi(\lambda)=0$

$\lambda^2-6\lambda+5=0$

$(\lambda-5)(\lambda-1)=0$ so $\lambda=1$ or 5.

- Next step, find the x's that go with each of these values for λ.

$\lambda=1$ find the x:

$A-\lambda I = A-I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

$\text{Nul}\left(\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}\right) = \left\{ x : \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \right\}$

set of solutions to $\boxed{Bx=0}$

Gaussian elimination to find x:

$\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2+2r_1} \begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix}$

pivot

free vbl

$\boxed{-3x_1} - 2x_2 = 0$

$x_1 = -\frac{2}{3}x_2$

$x_2 = x_2$

in vector form

$x = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} x_2$

which x do you choose? → any of these x's are eigenvectors (as long as they are not zero).

① Choose an x which is a unit vector.

② Choose an x which is easy to write down.

for now $x_2 = -3$

Thus $\lambda = 1$ and $x = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\lambda = 5$ Find the x:

keep this row

$$A - \lambda I = A - 5I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$$

Since $A - 5I$ has a free variable then the reduced row eschelon form has a zero row and a non-zero row... without any computation I can just keep one of the non-zero rows and cross the other out...

$$\underset{\text{pivot}}{2x_1} + \underset{\text{free vbl}}{4x_2} = 0 \qquad x_1 = -2x_2$$

in vector form

$$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

Thus $\lambda = 5$ and $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

In summary

eigenvalues	eigenvectors
$\lambda = 1$	$x = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$
$\lambda = 5$	$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Converted matrix vector mult to scalar vector mult...

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Applications:

1. Linear systems ~~of differential~~ equation.
2. Matrix factorization.

Matrix factorization

bunch them together.

factor the 1 and the 5 out

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} = \left[1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} \middle| \; 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

represents a column operation since on the right...

on the right

EA

row operation, matrix

$(EA)^T = A^T E^T$ on the left

column operation

Therefore

note that S is invertible

$$\underbrace{\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}}_{D}.$$

not so simple just a lin. function

diagonal matrix is very easy to understand...

$$AS = SD$$

$$AS S^{-1} = SDS^{-1}$$

Therefore $A = SDS^{-1}$ ← matrix factorization

although S is not an easy to understand matrix, this pattern with S^{-1} on the right and S on the left is easy to understand...

this pattern is called a conjugacy ...

$$A^2 = AA = SD\cancel{S^{-1}}\cancel{S}DS^{-1} = SD^2S^{-1}$$

cancelation when take powers of a matrix...

$$A^3 = AAA = SD\cancel{S^{-1}}\cancel{S}D\cancel{S^{-1}}\cancel{S}DS^{-1} = SD^3S^{-1}$$

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 1^2 & 0 \\ 0 & 5^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1^3 & 0 \\ 0 & 5^3 \end{bmatrix}$$

This power A^α

$$\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}^\alpha = SD^\alpha S^{-1} = S\begin{bmatrix} 1^\alpha & 0 \\ 0 & 5^\alpha \end{bmatrix}S^{-1}$$

makes sense for any value of α that makes sense on the right hand side

$\alpha = \sqrt{2}$ makes sense $\alpha = -1$ makes sense (inverse).

or square root $\alpha = 1/2$