Given:

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Let

Columns are eigenvectors of A

$$S = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

diagonal is the eigenvalues of A.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then

$AS = SD$ thus $\boxed{A = SDS^{-1}}$ factorization

mult on the right by S^{-1}

Some time ago we discussed taking powers of a matrix...

Powers of a Matrix

For the output of A to have a dimension suitable to use as the input, A must be squ

$$A^2 = AA$$

$$A^3 = AAA$$

$$\vdots$$

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}} = \prod_{i=1}^{k} A$$

With numbers (that is 1x1 matrices)

$$3^2 = 3 \cdot 3$$

$$3^{1/2} = \sqrt{3}$$

$$3^{p/q} = \sqrt[q]{3^p}$$

$$3^{\sqrt{2}} = e^{\sqrt{2}\ln 3}$$

What is $A^{1/2}$?

What is $A^{p/q}$?

What is $A^{\sqrt{2}}$?

$A = SDS^{-1}$

How can this factorization be used to answer these questions?

whats simple in this factorization is not S but where S appears in the factorization

$$A^2 = AA = SD\cancel{S^{-1}S}DS^{-1} = SDDS^{-1} = SD^2S^{-1}$$

this is great because D is a simple matrix

$$D^2 = DD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix}$$

$$D^3 = D^2D = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix}$$

$$A^2 = S\begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix}S^{-1}$$

$$A^3 = S\begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix}S^{-1}$$

$\vdots$

$$A^k = S\begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}S^{-1}$$

What is $A^{1/2}$? Guess...

$$A^{1/2} = S \begin{bmatrix} 1^{1/2} & 0 & 0 \\ 0 & 2^{1/2} & 0 \\ 0 & 0 & 3^{1/2} \end{bmatrix} S^{-1}$$

Check

$$A^{1/2} A^{1/2} = S \begin{bmatrix} 1^{1/2} & 0 & 0 \\ 0 & 2^{1/2} & 0 \\ 0 & 0 & 3^{1/2} \end{bmatrix} \cancel{S^{-1}} \cancel{S} \begin{bmatrix} 1^{1/2} & 0 & 0 \\ 0 & 2^{1/2} & 0 \\ 0 & 0 & 3^{1/2} \end{bmatrix} S^{-1}$$

$$= S \begin{bmatrix} 1^{1/2} & 0 & 0 \\ 0 & 2^{1/2} & 0 \\ 0 & 0 & 3^{1/2} \end{bmatrix} \begin{bmatrix} 1^{1/2} & 0 & 0 \\ 0 & 2^{1/2} & 0 \\ 0 & 0 & 3^{1/2} \end{bmatrix} S^{-1}$$

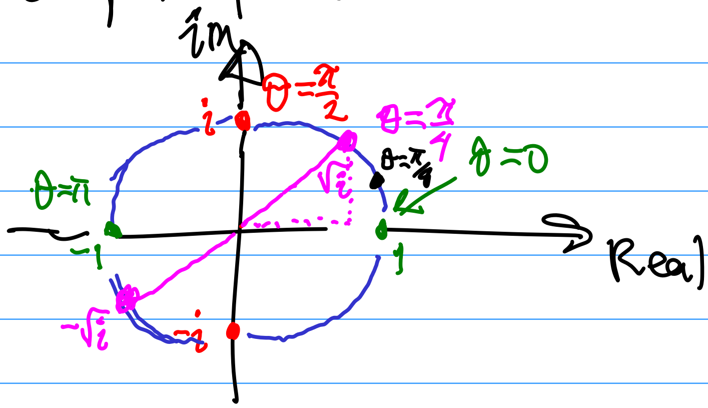
$$= S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1} = A$$

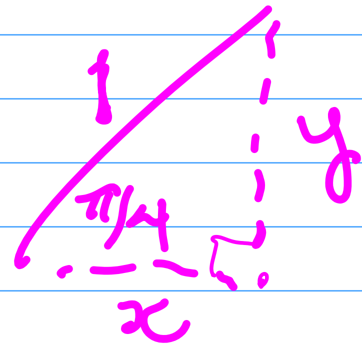
- Note not all square matrices can be factored as $A = SDS^{-1}$ where D is diagonal ... Then those matrices don't have square roots...

- What if the eigenvalues are negative? or even complex? Note that $\sqrt{-1} = i$ and $\sqrt{i} = \boxed{\quad}$

$(-i)(-i) = i^2 = -1$ so $\sqrt{-1} = \pm i$

Complex plane





$x = y$ $\quad x^2 + y^2 = 1$

$2x^2 = 1$

$x = \frac{1}{\sqrt{2}}$

$y = \frac{1}{\sqrt{2}}$

Therefore

$\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

and again

$$\sqrt{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = e^{i\pi/8}.$$

There is a type of matrix that guarantees the SDS^{-1} factorization exists and that the entries of D are real.

These are called the symmetric matrices
That is $\boxed{A^T = A}$ where $A \in \mathbb{R}^{n\times n}$.

This result is called the spectral theorem...

- A symmetric matrix where the entries of D are non-negative is called positive semi-definite.
- A symmetric matrix where the entries of D are strictly positive is called (strictly) positive definite.

strictly positive definite and positive definite will mean the same thing in this class...

Assume $A^T = A$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$A^T = A$ means $c = b$ if

Thus a symmetric 2x2 matrix look like

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$\chi(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix}$$

$$= (a-\lambda)(d-\lambda) - b^2$$

$$= \lambda^2 - (a+d)\lambda + ad - b^2$$

$\alpha = 1 \qquad \beta = -(a+d) \qquad \gamma = ad - b^2$

Quadratic formula...

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0$$

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

plug this in and check that

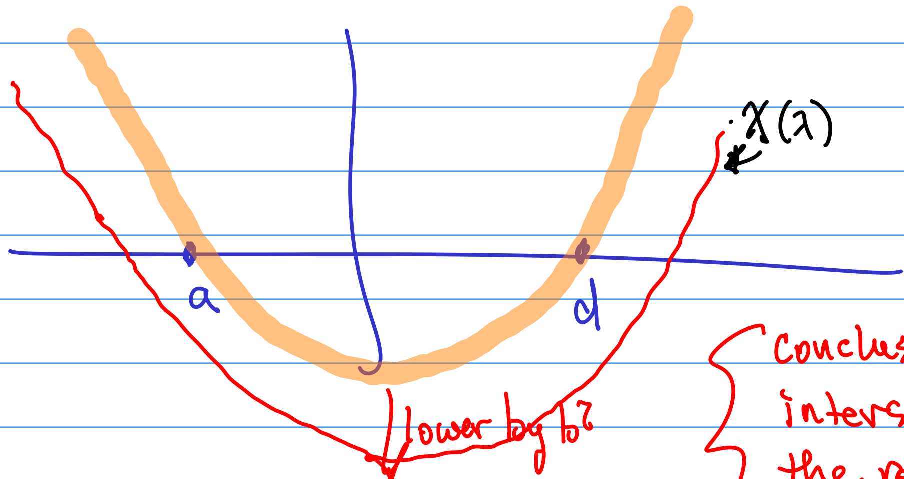
$$\boxed{\beta^2 - 4\alpha\gamma \geq 0}$$

Find roots of this:

$$\chi(\lambda) = \boxed{(a-\lambda)(d-\lambda)} - b^2$$

$$\chi(\lambda) = \boxed{(\lambda-a)(\lambda-d)} - b^2$$

this part in orange is shifted downwards by b^2 ...



Conclusion: the red curve intersects the x axis so the roots $\chi(\lambda)=0$ are real.

If $A^T = A$ then $\chi(\lambda)$ passes through the x-axis n times where $A \in \mathbb{R}^{n \times n}$ (counted by multiplicity).

If $Ax = \lambda x$ and A is real what does that mean about λ. If you think about λ as a complex number the imaginary part is zero.

Example: $2+5i$ complex conjugate is $2-5i$

Notation

$$\overline{2+5i} = 2-5i$$

or in another book

$$(2+5i)^* = 2-5i$$

$$(2+5i)^\dagger = 2-5i$$

notations we will not use that mean the same thing

$Ax = \lambda x$ and $A^T = A$

what does it mean for λ to be real?

$$\bar{\lambda} = \lambda$$