5.5

Foil method to multiply two complex numbers.

$$i^2 = -1$$

$$(2+3i)(1-2i) = 2 - 4i + 3i - 6i^2$$

Conjugates

$$= 2 - i + 6 = 8 - i$$

$$(2-3i)(1+2i) = 2 + 4i - 3i - 6i^2$$

Conjugate

$$= 2 + i + 6 = 8 + i$$

In particular if $w, z \in \mathbb{C}$ set of complex numbers.

Then $\overline{wz} = \bar{w}\,\bar{z}$

What is a complex vector?

$x \in \mathbb{C}^n$ a complex $n$-vector...

Example $n=3$

$$x = \begin{bmatrix} 1+3i \\ -2+5i \\ 1-7i \end{bmatrix} \qquad \bar{x} = \overline{\begin{bmatrix} 1+3i \\ -2+5i \\ 1-7i \end{bmatrix}} = \begin{bmatrix} 1-3i \\ -2-5i \\ 1+7i \end{bmatrix}$$

In general when solving the eigenvalue-eigenvector problem we find $\lambda$ by $\det(A-\lambda I) = 0$ which is a polynomial equation of degree $n$.

Given complex values of $\lambda$ what can I do with them?

$$\text{Nul}(A-\lambda I) = \{x : (A-\lambda I)x = 0\}$$

and I find non-zero eivectors $x$ that correspond to eigenvalues... We can find the eigenvectors using Gaussian elimination just like before... in particular, if $\det(A-\lambda I)=0$ then there are free variables and so there are eigenvectors...

---

$a, b \in \mathbb{R}$

Suppose $\lambda \in \mathbb{C}$ of the form $\lambda = a+ib$ where $b \neq 0$.

Then the eigenvectors $x$ are always complex.

assuming $A \in \mathbb{R}^{n\times n}$

Notation: $\text{Re}\,\lambda = a$ and $\text{Im}\,\lambda = b$

$x \in \mathbb{C}^n$ and $x = u + iv$ where $u, v \in \mathbb{R}^n$

Thus $\text{Re}\,x = u$ and $\text{Im}\,x = v$

---

Consider the 2x2 case,,, $n=2$, $A \in \mathbb{R}^{2\times 2}$.

$$Ax = A(u+iv) = Au + iAv$$

$$\lambda x = (a+ib)(u+iv) = au + iav + ibu - bv$$

$$= au - bv + i(av+bu)$$

Thus

$$Au = au - bv \quad \text{and} \quad Av = av + bu$$

$$A\left[u \mid v\right] = \left[Au \mid Av\right] = \left[au-bv \mid av+bu\right]$$

$$= \left[u \mid v\right]\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

---

Let $P = \left[u \mid v\right] = \left[\operatorname{Re} x \mid \operatorname{Im} x\right]$

Then

$$AP = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\boxed{A = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}P^{-1}}$$

like a rotatation

not diagonal but at least not complex...

$\lambda = a + ib$

$Ax = \lambda x$

$\overline{Ax} = \overline{\lambda x}$

$A\bar{x} = \bar{\lambda}\bar{x}$

So $\bar{\lambda}$ is also an eigenvalue and $\bar{x}$ is the corresponding eigenvector...

---

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \qquad S = \left[x \mid \bar{x}\right]$$

$$\boxed{A = S\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}S^{-1}}$$

complex matrix but diagonal.

Spectrum of a matrix is the set of eigenvalues...

Spectral theorem says if $A^T = A$ then the eigenvalues are all real numbers... (and more).

Hypothesis

$A^T = A$ what does this say about $\lambda$ and why?

If $\lambda \in \mathbb{R}$ this means $\lambda = \bar{\lambda}$.

Now consider... since $Ax = \lambda x$ ...

$$\bar{x} \cdot Ax = \bar{x} \cdot \lambda x = \lambda\, \bar{x} \cdot x$$

$$\|$$

$$Ax \cdot \bar{x} = (Ax)^T \bar{x} = x^T A^T \bar{x} = x^T A \bar{x} = x \cdot A\bar{x}$$

$$= x \cdot \bar{\lambda} \bar{x}$$

$$= \bar{\lambda}\, x \cdot \bar{x}$$

$$= \bar{\lambda}\, \bar{x} \cdot x$$

recall this

$Ax = \lambda x$

$\overline{Ax} = \overline{\lambda x}$

$A\bar{x} = \bar{\lambda} \bar{x}$

So $\bar{\lambda}$ is also an eigenvalue and $\bar{x}$ is the corresponding eigenvector...

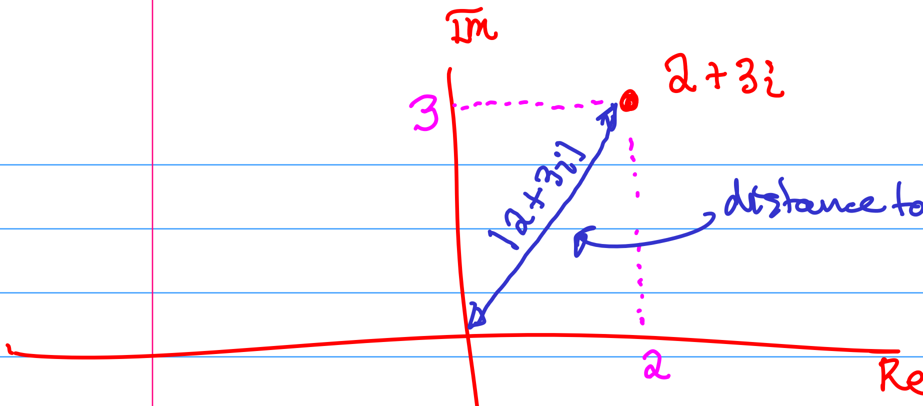
Therefore

$$\lambda \cancel{\bar{x} \cdot x} = \bar{\lambda} \cancel{\bar{x} \cdot x}$$

$$\lambda = \bar{\lambda}$$

So $\lambda$ is real.

☑ Need to make sure $\bar{x} \cdot x \neq 0$ so the cancellation makes sense.




distance to the origin $= \sqrt{2^2+3^2}$

$= \sqrt{4+9} = \sqrt{13}$

$\left\{\begin{array}{c}\text{modulus}\\ \text{absolute value}\\ \text{norm}\end{array}\right\}$ is the distance to the origin

$$(2+3i)(2-3i) = 4 - \cancel{6i} + \cancel{6i} - 9i^2 = 4+9 = 13$$

$w = 2+3i$ then $|w| = \sqrt{w\overline{w}}$ or $|w|^2 = w\overline{w}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \overline{x} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{bmatrix}$$

$$x \cdot \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{bmatrix} = x_1\overline{x_1} + x_2\overline{x_2} + \cdots + x_n\overline{x_n}$$

$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

Since $x \neq 0$ then at least one of these are non-zero and so $x \cdot \overline{x} = \|x\|^2 > 0$.