Chapter 6.1

Pythagorean theorem in n-dimensional space..

for $v \in \mathbb{R}^n$ then $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

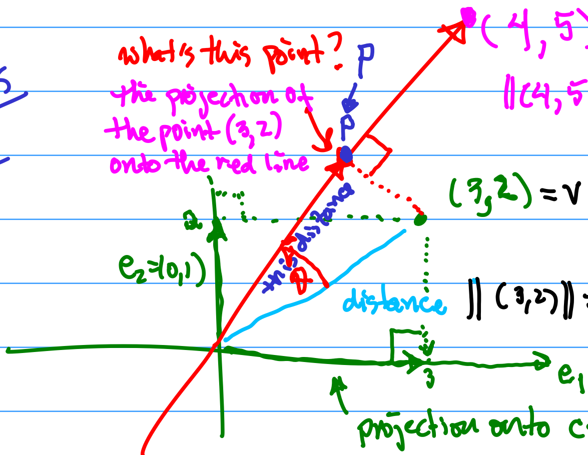
distance from the origin..

the the square of a real number is always positive..

for $x \in \mathbb{C}^n$ then $\|x\| = \sqrt{x \cdot \bar{x}} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

since that's not true for complex numbers, we throw in an absolute value as well..

Examples
$n = 2$



$\|(4,5)\| = \sqrt{4^2 + 5^2}$

$\|(3,2)\| = \sqrt{3^2 + 2^2}$

projection onto coordinate axis... can write these projections in terms of dot products... and the fact that e_1 and e_2 are unit vectors...

$3 = (3,2) \cdot (1,0) = 3 \cdot 1 + 2 \cdot 0 = 3$

$2 = (3,2) \cdot (0,1) = 3 \cdot 0 + 2 \cdot 1 = 2$

note if you rescale e_1 so it's not a unit vector, then the dot product doesn't give the coordinates...

Idea: Take the unit vector in the u direction and dot it into $(3,2)$ to find the distance in the u direction of the projection..

unit vector in the same direction as u

since $(4,5) = u$ then $\hat{u} = \dfrac{u}{\|u\|}$

$$\|\hat{u}\| = \sqrt{\hat{u} \cdot \hat{u}} = \sqrt{\frac{u}{\|u\|} \cdot \frac{u}{\|u\|}} = \sqrt{\frac{u \cdot u}{\|u\|^2}} = \frac{\|u\|}{\|u\|} = 1.$$

$$\boxed{\|p\| = \hat{u} \cdot v = \frac{u}{\|u\|} \cdot v = \frac{(4,5) \cdot (3,2)}{\|(4,5)\|} = \frac{4 \cdot 3 + 5 \cdot 2}{\sqrt{41}} = \frac{22}{\sqrt{41}}}$$

Then what is p itself?

$$p = \|p\| \, \hat{u} = \frac{22}{\sqrt{41}} \, \frac{(4,5)}{\sqrt{41}} = \frac{22}{41}(4,5)$$

Interpretation of a dot product in terms of the angle between the vectors...

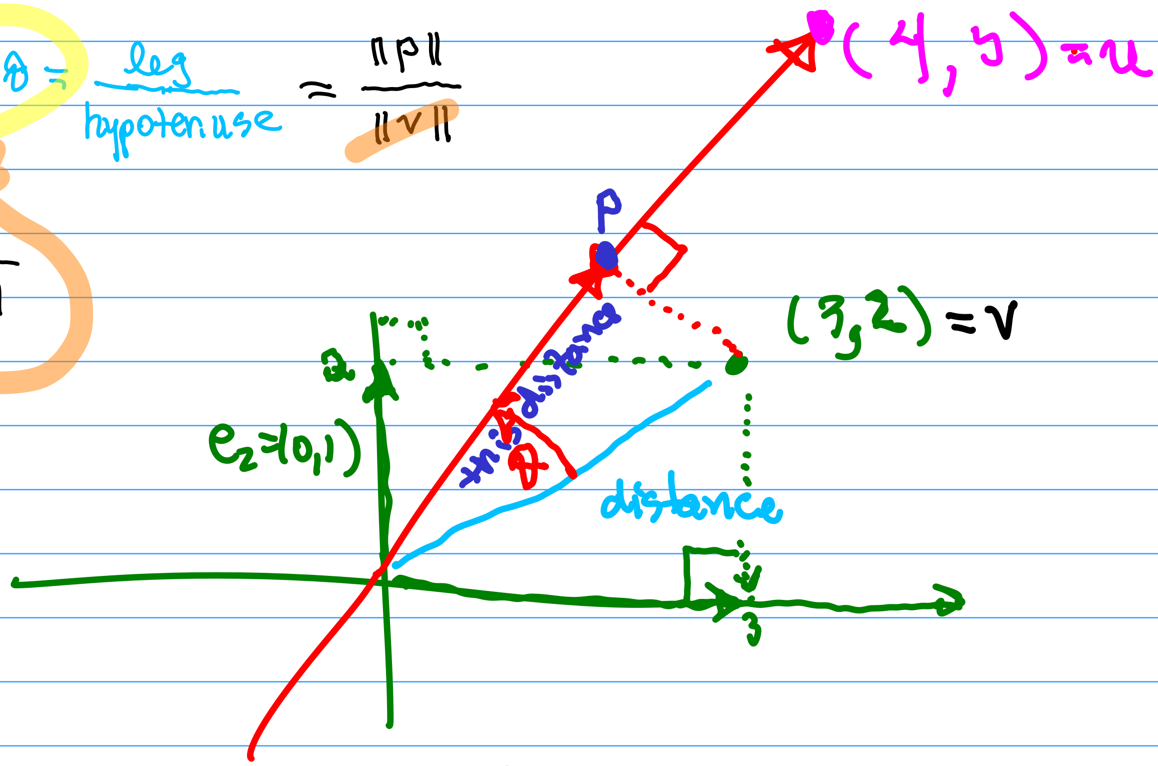
$$u \cdot v = \|u\| \, \|v\| \cos\theta$$

I think about this as simply the definition of θ.

where θ is the angle between u and v.

$$\frac{u \cdot v}{\|u\|\|v\|} = \cos\theta = \frac{\text{leg}}{\text{hypotenuse}} = \frac{\|p\|}{\|v\|}$$

$$\|p\| = \frac{u \cdot v}{\|u\|}$$



Gram-schmidt orthogonalization...

$$A = \begin{bmatrix} x_1 \,|\, x_2 \,|\, \dots \,|\, x_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

and columns of A are linearly independent... Thus $Az = 0$ has only the unique solution $z = 0$... that is there are no free variables

Note, this is a column operation.. a rescaling operation

$u_1 = x_1$ $\qquad$ $v_1 = \dfrac{u_1}{\|u_1\|}$

$u_2 = x_2 - (v_1 \cdot x_2) v_1$ $\qquad$ $v_2 = \dfrac{u_2}{\|u_2\|}$

Claim is that $v_1 \cdot v_2 = 0$.

$$v_1 \cdot v_2 = v_1 \cdot \frac{u_2}{\|u_2\|} = \frac{1}{\|u_2\|}(v_1 \cdot u_2)$$

$$= \frac{1}{\|u_2\|} v_1 \cdot \left(x_2 - (v_1 \cdot x_2) v_1\right) = \frac{1}{\|u_2\|}\left(\cancel{v_1 \cdot x_2} - \cancel{(v_1 \cdot x_2)} \underbrace{(v_1 \cdot v_1)}_{=1}\right)$$

since v_1 is a unit vector

everthing cancels so $v_1 \cdot v_2 = 0$..

$u_3 = x_3 - (v_1 \cdot x_3) v_1 - (v_2 \cdot x_3) v_2$ $\qquad$ $v_3 = \dfrac{u_3}{\|u_3\|}$

subtract off the part of x_3 that's in the v_2 direction

subtract of the part of x_3 that's in the v_1 direction...

The result is that u_3 is perpendicular to both v_1 and v_2.

$u_4 = x_4 - (v_1 \cdot x_4) v_1 - (v_2 \cdot x_4) v_2 - (v_3 \cdot x_4) v_3$ $\qquad$ $v_4 = \dfrac{u_4}{\|u_4\|}$

column operations on A

$\vdots$

$u_n = x_n - (v_1 \cdot x_n) v_1 - \cdots - (v_{n-1} \cdot x_n) v_{n-1}$ $\qquad$ $v_n = \dfrac{u_n}{\|u_n\|}$

Now save the columns as the new matrix

$$Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Eventually we'll put the results of the column operations into another matrix called R so that $A = QR$... (not today)...

triangular matrix..

R is nice because it's triangular.

What's nice about Q?

$$Q = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$v_1 \cdot v_1 = 1 \quad v_1 \cdot v_2 = 0$

$v_2 \cdot v_1 = 0 \quad v_2 \cdot v_2 = 1$

$$Q^T Q = \begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix} = I$$

If Q were square this would imply that $Q^{-1} = Q^T$

If Q is not square it's still nice that $Q^T Q = I$.