obtain Q using column operations on A
That's called the Gram-Schmidt process...

$$A = QR$$

(also square and invertible)

upper triangular

satisfies $Q^TQ = I$ use this

Has orthonormal columns

, this
stems
ne

Let $A \in \mathbb{R}^{m \times n}$

R is invertible (and square)

$$\mathrm{Col}\, A = \{ Ax : x \in \mathbb{R}^n \} = \{ Q(Rx) : x \in \mathbb{R}^n \}$$

$y = Rx$ then for every $y \in \mathbb{R}^n$
then setting $x = R^{-1}y$ yields

$$Rx = RR^{-1}y = y$$

Therefore

$$\mathrm{Col}\, A = \{ Qy : y \in \mathbb{R}^n \} = \mathrm{col}\, Q$$

THEOREM 2 **The Pythagorean Theorem**

Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

u + v

$u \perp v$ means u is perpendicular to v

means $u \cdot v = 0$

In general anything with dot products can be written using norms and vice versa...

$\|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + u\cdot v + v\cdot u + v\cdot v$

$= u\cdot u + 2u\cdot v + v\cdot v =$

$= \|u\|^2 + 2u\cdot v + \|v\|^2$

$u\cdot v = \frac{1}{2}\left(\|u+v\|^2 - \|u\|^2 - \|v\|^2\right)$ form of the polarization identity...

means this is zero

remark, this is how to check if you have a right triangle in geometry $a^2+b^2=c^2$ means its a right triangle

Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$

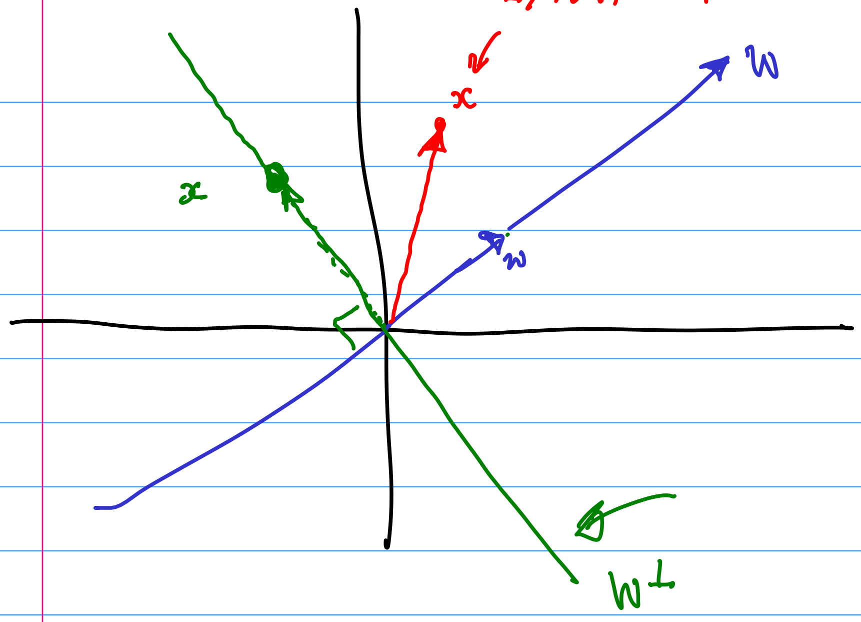
row A is the span of the rows of A

Note row A = col A^T

Let W be a subspace then $W^{\perp}$ is the orthogonal complement of W

$$\boxed{W^{\perp} = \{x : x\cdot w = 0 \text{ for all } w \in W\}}$$

Is this x perpendicular to all vectors in W? No



Again.
$w \in W$ and $x \in W^\perp$
then $w \cdot x = 0$.

Back to the theorem

$(\text{Col } A)^\perp = \text{Nul } A^T$

$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \}$

$w \in \text{Col } A$ then $w = Az$ for some $z \in \mathbb{R}^n$

$W^\perp = \{ x : x \cdot w = 0 \text{ for all } w \in W \}$

$(\text{Col } A)^\perp = \{ x : x \cdot w = 0 \text{ for all } w \in \text{Col } A \}$

$= \{ x : x \cdot Az = 0 \text{ for all } z \in \mathbb{R}^n \}$

Now

$x \cdot Az = x^T Az = (A^T x)^T z = A^T x \cdot z$

$(\text{Col } A)^\perp = \{ x : A^T x \cdot z = 0 \text{ for all } z \in \mathbb{R}^n \}$.

Note if $A^T x \cdot z = 0$ for all z then $A^T x = 0$.

for example, take $z = A^T x$ since z can be anything.
then $A^T x \cdot z = A^T x \cdot A^T x = \| A^T x \|^2$ so $\| A^T x \| = 0$ so $A^T x = 0$.

$(\text{Col } A)^\perp = \{ x : A^T x = 0 \} = \text{Nul } A^T$

Reflect upon why this worked...

$$x \cdot Az = A^T x \cdot z$$ ← main idea....

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$ ☑

?

substitute A^T for A

$\text{Row } A = \text{Col } A^T$

$(\text{Col } A^T)^{\perp} = \text{Nul } A^{TT} = \text{Nul } A$

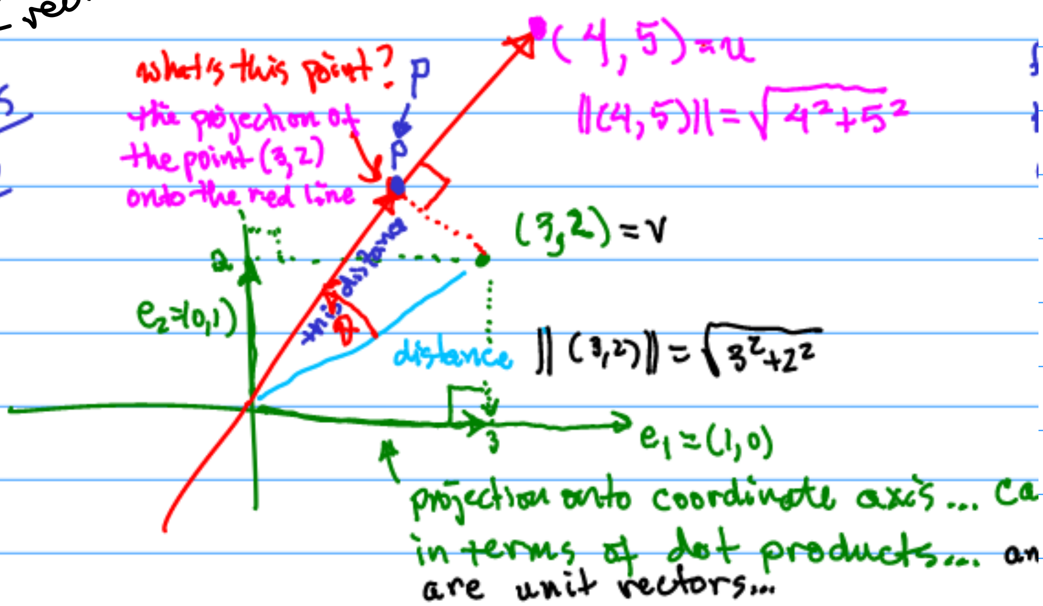
$(\text{Row } A)^{\perp} = \text{Nul } A$ ← so this is the same result as the other one...

$\boxed{\text{Now what about 6.2}}$

From before we were projection onto another vector

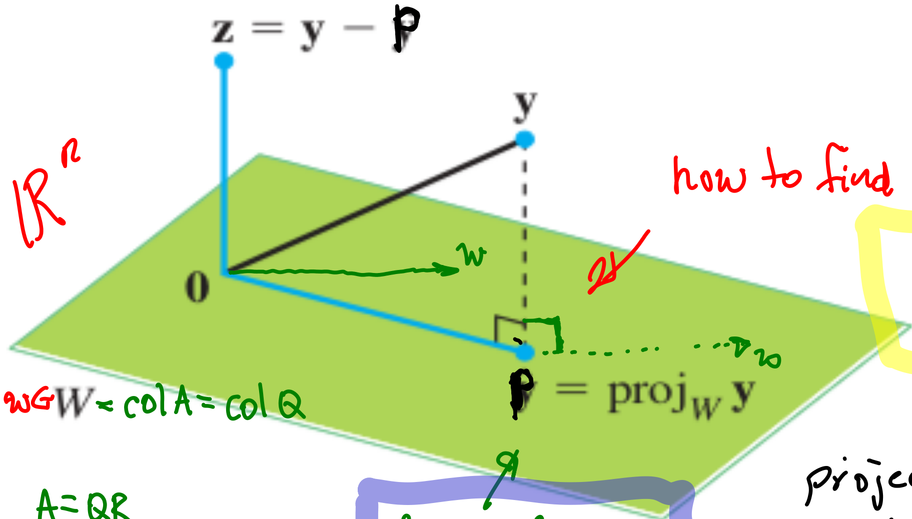
Examples

$n=2$



Now, I want to project onto a subspace...

$\hat{y}$ is not a unit vector. Huh?



$\mathbb{R}^n$

$w \in W = \text{col}\,A = \text{col}\,Q$

$A = QR$

how to find p?

$\boxed{p - y \perp w \text{ for every } w \in W}$

$\boxed{\text{formula for } p: \; p = QQ^Ty}$

Projecting onto a plane rather than a line...

rather than deriving this formula from the condition I'll just check it works.

FIGURE 2
Finding α to make $y - \hat{p}$ orthogonal to **u**.

W is a subspace, so W has a basis $\{b_1, b_2, \ldots, b_q\}$ where $\dim W = q$ and $W = \text{col}\,A$

$$A = \left[b_1 \,|\, b_2 \,|\, \cdots \,|\, b_q \right] \in \mathbb{R}^{n \times q}$$

make QR factorization of A using Gram-Schmidt...

$$\underbrace{Q = \left[v_1 \,|\, v_2 \,|\, \cdots \,|\, v_q \right]}_{\text{orthogonal}} \in \mathbb{R}^{n \times q} \qquad \underbrace{R = \begin{bmatrix} \|u_1\| & v_1 \cdot b_2 & \cdots & v_1 \cdot b_q \\ 0 & \|u_2\| & \ddots & \vdots \\ & & & v_{q-1} \cdot b_q \\ 0 & 0 & \ddots & \|u_q\| \end{bmatrix}}_{\text{invertibe}} \in \mathbb{R}^{q \times q}$$

$A = QR \ldots$

$W = \text{Col}\, A = \text{col}\, Q$

$p = QQ^T y$

Check this

$\boxed{p - y \perp w \text{ for every } w \in W}$ & need $(p-y) \cdot w = 0$ for every $w \in W$.

If $w \in W$ then $w \in \text{col}\, Q$ which means $w = Qx$ for some x.

$(p - y) \cdot w = 0$

need to show this...

$(QQ^T y - y) \cdot Qx = 0$ for all x

$\boxed{QQ^T y \cdot Qx} - \cancel{y \cdot Qx} = 0$

simplify this

since columns of Q are orthonormal then $Q^T Q = I$

$QQ^T y \cdot Qx = (QQ^T y)^T Qx = y^T Q^{TT} \cancel{Q^T} \cancel{Q} x = y^T Qx$

$= \cancel{y \cdot Qx}$

- Therefore $p = QQ^T y$ is the orthogonal projection of the point y onto the subspace W.