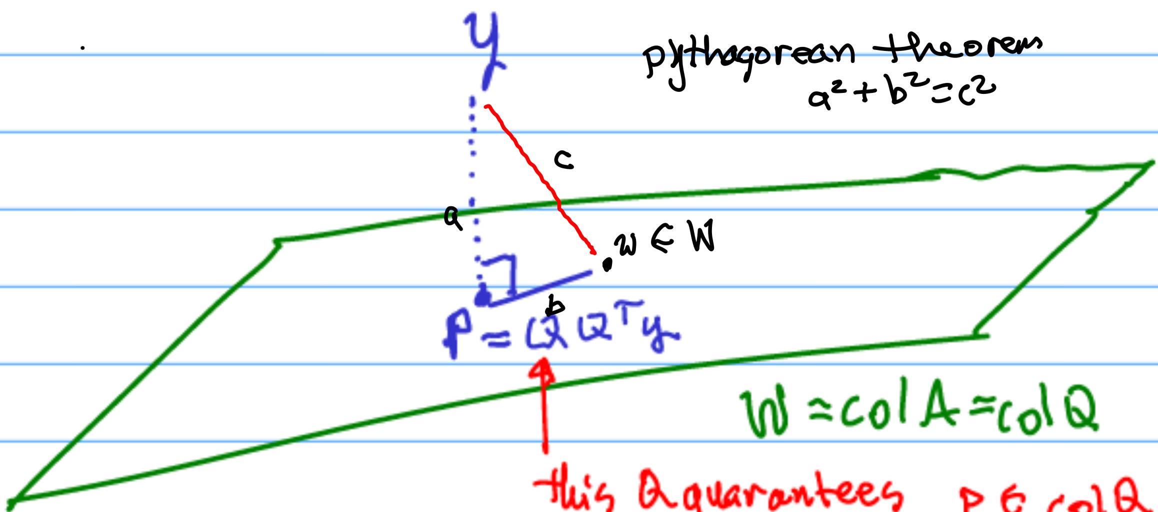
$p = QQ^Ty$ Check this...



Claim that the point given by the orthogonal projection is the closest point in W to the point y.

Suppose $w \in W$ when w is not perpendicular to $p-y$.
Then $w \neq p$. Then can make a triangle... pythagorean theorem

$$\|p-y\|^2 + \|p-w\|^2 = \|y-w\|^2$$

$p = QQ^Ty$

$$\|p-y\|^2 = \|y-w\|^2 - \|p-w\|^2 < \|y-w\|^2$$

The distance between p and y is strictly smaller than the distance between y and any other point in W.

Theorem 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$

- Note $\text{Row}\,A = \text{Col}\,A^T$ just by definition, so these two equalities say exactly the same thing except with A^T substituted for A

$$W^{\perp} = \{ y : y \cdot w = 0 \text{ for all } w \in W \}.$$

If $A \in \mathbb{R}^{m \times n}$ then

$$\text{Col}\,A = \{ Ax : x \in \mathbb{R}^n \}$$

$$(\text{Col}\,A)^{\perp} = \{ y : y \cdot w = 0 \text{ for all } w \in \text{Col}\,A \}$$

$$= \{ y : y \cdot Ax = 0 \text{ for all } x \in \mathbb{R}^n \}$$

when A jumps over the dot it gets a transpose...

remember

$$y \cdot Ax = y^T A x = (A^T y)^T x = A^T y \cdot x$$

$$(\text{Col}\,A)^{\perp} = \{ y : A^T y \cdot x = 0 \text{ for all } x \in \mathbb{R}^n \}$$

Note if $v \cdot x = 0$ for all $x \in \mathbb{R}^n$

this means $v = 0$. (duality argument).

Why? Take $x = v$ then $v \cdot x = v \cdot v = \|v\|^2 = 0$

this means $v = 0$.

Therefore...

$$(\text{Col}\,A)^{\perp} = \{ y : A^T y = 0 \} = \text{Nul}(A^T).$$

that's the result of Theorem 3...

Use QR to solve a minimization problem...

From last Thursday...

$$\underbrace{\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}}_{A} = \underbrace{\frac{1}{2\sqrt{3}}\begin{bmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2\sqrt{3} & -6\sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{3} & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}}_{R}$$

Want to solve...

$$Ax = b$$

note since A has more rows than columns this system is overdetermined... that mean's it's inconsistent unless something special happens...

4 equations

3 unknowns

$A : \mathbb{R}^3 \to \mathbb{R}^4$

$A \in \mathbb{R}^{4\times 3}$

If this can be solved that means $b \in \operatorname{col} A$ ... but since there are more rows than columns then col A is a subspace ...

Note $\dim \operatorname{col} A = 3$ since there are 3 pivots in A. Since the dimension of $\mathbb{R}^4$ is 4 then Col A is a strict subspace.

When $Ax=b$ has no solution, the next best thing is to minimize the error represented by

$$E = \| Ax - b \|$$

So I'm looking for the value of x such that Ax is as close as possible to b.

$Ax \in \text{Col } A = W$

so what is the closest point in W to b?

The orthogonal projection of b onto W.

same

$p = QQ^Tb$

So

$Ax = p = QQ^Tb$

this has a solution... we can find it using the QR decomposition just like before..

$A = QR$

$QRx = QQ^Tb$

$\cancel{Q^T}QRx = \cancel{Q^T}QQ^Tb$

$Rx = Q^Tb$ ← same equation from last time that we used to solve $Ax=b$ when it did have a solution...

So, whether $Ax=b$ has a solution or not, we do the same thing either way and find x by solving $Rx = Q^Tb$.

$\boxed{\text{Starting } 6.5}$ ← actualy ends with this equation for finding the least square solution to $Ax=b$, that is to minimize $\|Ax-b\|$.

6.5 starts with the normal equations, which are just rewriting $Rx = Q^T b$ in terms of A.

Since $A = QR$ (Q has orthonormal columns; R square, invertible, triangular) whats this in terms of A?

First consider

$$A^TA = (QR)^T QR = R^T Q^T Q R = R^T R$$

R^T invertible, R invertible; product is invertible...

$Q^TQ = I$ because Q has orthonormal columns

Therefore A^TA is invertible...

- If A has linearly independent columns then A^TA is invertible... (T/F)

$Rx = Q^T b$

$A = QR$

$AR^{-1} = Q$

$Q^T = (AR^{-1})^T = (R^{-1})^T A^T = (R^T)^{-1} A^T$

Thus...

$Rx = (R^T)^{-1} A^T b$

other side

$R^T R x = A^T b$

- rewrote $Rx = Q^T b$ in terms of A...

$$A^T A x = A^T b$$

Normal equations for solving the least squares problem $Ax = b$

when A is very small matrix this could be practical, but usually its better to use QR instead.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

A Q R

Find the x which minimizes $\|Ax-b\|$...

$Rx = Q^T b$

$$Q^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

Solve this...

$$\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

$2x_1 + 3x_2 = 17/2$

$5x_2 = 9/2$

back substitution to solve

$x_2 = \frac{9}{10}$

$3 \cdot \frac{17}{5}$ / 85

$$x_1 = \frac{17/2 - 3x_2}{2} = \frac{\frac{17}{2} - \frac{27}{10}}{2} = \frac{85 - 27}{20} = \frac{58}{20} = \frac{29}{10}$$

Answer $x = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$ minimizes $\|Ax-b\|$.

Example test question

Solve $Ax = b$ where $A = QR$ and $Q = [\ ]$ and $R = [\ ]$.

Do it again using the normal equations

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17 \\ 48 \end{bmatrix}$$

$$\begin{array}{r|r} 1 & \\ 24 & 25 \\ -5 & 20 \\ 28 & 48 \end{array}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}$$

Solve

$$\begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 48 \end{bmatrix}$$

To save time let's just check that

$$x = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}$$

is a solution...

Check

$$\frac{2}{10} \begin{bmatrix} 2 & 3 \\ 3 & 17 \end{bmatrix} \begin{bmatrix} 29 \\ 9 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 85 \\ 240 \end{bmatrix} = \begin{bmatrix} 17 \\ 48 \end{bmatrix} \boxed{\checkmark}$$

$$\begin{array}{r} {}^{1}\\ 58 \\ +\,27 \\ \hline 85 \end{array} \qquad \begin{array}{r} {}^{2}\\ 29 \\ 3 \\ \hline 87 \end{array} \qquad \begin{array}{r} {}^{6}\\ 17 \\ 9 \\ \hline 153 \end{array} \qquad \begin{array}{r} {}^{1}\\ 153 \\ +\,87 \\ \hline 240 \end{array} \qquad \begin{array}{r} {}^{4}\\ 48 \\ 5 \\ \hline 240 \end{array}$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\|v\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$